Math 4650 Homework #9 Solutions

1. 5.1, #9bc. *Picard's method is described as follows: Let $y_0(t) = \alpha$ for each $t \in [a, b]$. Define a sequence $\{y_k(t)\}$ of functions by*

$$y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))\, d\tau, \qquad k = 1, 2, \ldots.$$

- *9b. Generate $y_0(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$ for the initial-value problem*

$$y' = -y + t + 1, \quad 0 \le t \le 1, \quad y(0) = 1.$$

$$\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t -1 + \tau + 1\, d\tau = 1 + \tfrac{1}{2}t^2 \\
y_2(t) &= 1 + \int_0^t -(1 + \tfrac{1}{2}\tau^2) + \tau + 1\, d\tau = 1 + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 \\
y_3(t) &= 1 + \int_0^t -(1 + \tfrac{1}{2}\tau^2 - \tfrac{1}{6}t^3) + \tau + 1\, d\tau = 1 + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4.
\end{aligned}$$

- *Compare the result in part (b) to the Maclaurin series of the actual solution $y(t) = t + e^{-t}$.*

  The Maclaurin series of the solution is

$$y(t) = t + 1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \tfrac{1}{24}t^4 - \tfrac{1}{120}t^5 + \cdots,$$

  and clearly the terms $y_k(t)$ agree completely with $y(t)$ up to the power t^{k+1}.

2. 5.2, #1b. *Use Euler's method to approximate the solution for the following initial-value problem.*

$$y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1, \text{ with } h = 0.5$$

The basic setup is $w_0 = 1$, $t_0 = 2$, $t_1 = 2.5$, and $t_2 = 3$. We want $w_2 \approx y(t_2) = y(3)$.

$$\begin{aligned}
w_1 &= w_0 + 0.5f(t_0, w_0) = w_0 + 0.5(1 + (t_0 - w_0)^2) = 1 + 0.5(1 + (2 - 1)^2) = 2. \\
w_2 &= w_1 + 0.5(1 + (t_1 - w_1)^2) = 2 + 0.5(1 + (2.5 - 2)^2) = 2 + 0.5(1.25) = 2.625.
\end{aligned}$$

So $y(3) \approx 2.625$.

3. 5.2, #3b. *The actual solution to the initial-value problem in Exercise 1 is given here. Compare the actual error at each step to the error bound.*

$$y(t) = t + \frac{1}{1 - t}.$$

We need the Lipschitz constant for the function and the maximum of the second derivative of the solution. By Theorem 5.3, the easiest way to get the Lipschitz constant is as $L = \max\{|\frac{\partial f}{\partial y}|\}$. Clearly $\frac{\partial f}{\partial y} = -2(t-y)$.

If we plug in the actual solution here, we get

$$\left|\frac{\partial f}{\partial y}(t, y(t))\right| = \left|2(t-(t+\tfrac{1}{1-t}))\right| = \left|\frac{2}{1-t}\right|,$$

and the maximum of this on the interval $2 \le t \le 3$ is 2.

That means any number strictly larger than 2 will work as a Lipschitz constant for the differential equation with these conditions. (We need strictly larger because we want a little room for error in y.)

Now the second derivative is $y''(t) = \frac{2}{(1-t)^3}$, and the maximum of this on $[2, 3]$ occurs at $t = 2$, so that $M = 2$. With $h = 0.5$, the theoretical error from equation (5.10) is

$$\frac{hM}{2L}(e^{L(t_i-a)} - 1) = \frac{1}{4}(e^{2(t_i-2)} - 1).$$

For $t_i = 2.5$ we have theoretical error $\varepsilon_1 = \frac{1}{4}(e^1 - 1) = 0.43$, while the actual error is $2 - 1.83 = 0.17$.

For $t_i = 3$ the theoretical error is $\varepsilon_2 = \frac{1}{4}(e^2 - 1) = 1.6$, while the actual error is $2.625 - 2.5 = 0.125$.

4. 5.2, #12. *Consider the initial-value problem*

$$y' = -10y, \quad 0 \le t \le 2, \quad y(0) = 1,$$

which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

Euler's method for this equation is

$$w_{i+1} = w_i + hf(t_i, w_i) = w_i - 10hw_i = 0.$$

So no matter what the initial condition is, Euler's method will predict zero for the first and every subsequence step.

When trying to apply Theorem 5.9, we have $L = 10$ and $M = \max_{0 \le t \le 2}\{100e^{-10t}\} = 100$. So the theoretical error is

$$\frac{hM}{2L}(e^{Lt_i} - 1) = \frac{1}{2}(e^{10t_i} - 1).$$

If $t_i = hi = 0.1i$, then the theoretical error is

$$0.5(e^i - 1).$$

The actual error is e^{-10t_i} for each t_i, or e^{-i}. Now for $i \ge 1$ we have

$$e^{-i} \le 0.5(e^i - 1),$$

so the actual error is consistent with the theoretical error.

5. 5.3 #1b. *Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.*

$$y' = 1 + (t-y)^2, \quad 2 \le t \le 3, \quad y(2) = 1, \text{ with } h = 0.5$$

The function is $f(t,y) = 1 + (t-y)^2$, so that $f_t(t,y) = 2(t-y)$ and $f_y(t,y) = -2(t-y)$. So Taylor's method says

$$w_{i+1} = w_i + hf(t_i, w_i) + \tfrac{1}{2}h^2\big[f_t(t_i, w_i) + f(t_i, w_i)f_y(t_i, w_i)\big],$$

or specifically

$$\begin{aligned} w_{i+1} &= w_i + h[1 + (t_i - w_i)^2] + \tfrac{1}{2}h^2\big[2(t_i - w_i) - 2(t_i - w_i)(1 + (t_i - w_i)^2\big) \\ &= w_i + h[1 + (t_i - w_i)^2] - h^2(t_i - w_i)^3. \end{aligned}$$

So with $w_0 = 1$, we have

$$w_1 = 1 + 0.5(1 + (2-1)^2) - 0.25(2-1)^3 = 1.75$$

and

$$w_2 = 1.75 + 0.5(1 + (2.5 - 1.75)^2) - 0.25(2.5 - 1.75)^3 = 2.43.$$

6. 5.4 #1b. *Use the modified Euler method to approximate the solution to the following initial-value problem, and compare the results to the actual values.*

$y' = 1+(t-y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ *with* $h = 0.5$*; actual solution* $y(t) = t + 1/(1-t)$.

The modified Euler method is

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))].$$

In this case we have $f(t_0, w_0) = f(2, 1) = 2$ so that $f(t_1, w_0 + hf(t_0, w_0)) = f(2.5, 2) = 1.25$. So

$$w_1 = 1 + 0.25(2 + 1.25) = 1.8125.$$

Then $f(t_1, w_1) = f(2.5, 1.8125) = 1.4727$, and $f(t_2, w_1 + hf(t_1, w_1)) = f(3, 2.5488) = 1.2036$. Thus

$$w_2 = 1.8125 + 0.25(1.4727 + 1.2036) = 2.4816.$$

The error in w_1 is 0.02, and so is the error in w_2.

7. 5.4, #5b. *Repeat exercise 1 using Heun's method.*

Heun's method is

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i + \frac{2}{3}h, w_i + \frac{2}{3}hf(t_i, w_i)\right)\right].$$

Again we have $f(t_0, w_0) = f(2, 1) = 2$, and thus

$$f(t_0 + \tfrac{2h}{3}, w_0 + \tfrac{2h}{3} f(t_0, w_0)) = f(2.3333, 1.6667) = 1.4444.$$

So

$$w_1 = 1 + 0.125(2 + 3 \cdot 1.4444) = 1.7917.$$

Then $f(t_1, w_1) = f(2.5, 1.7917) = 1.5017$, so that

$$f(t_1 + \tfrac{2h}{3}, w_1 + \tfrac{2h}{3} f(t_1, w_1)) = f(2.8333, 2.2922) = 1.2928.$$

Thus finally

$$w_2 = 1.7917 + 0.125(1.5017 + 3 \cdot 1.2928) = 2.4642.$$

The errors in w_1 and w_2 are both 0.04.

8. 5.4, #9b. *Repeat exercise 1 using the midpoint method.*

   The midpoint method says

$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right).$$

   We have $f(t_0, w_0) = f(2, 1) = 2$ again, so that

$$w_1 = w_0 + hf(t_0 + 0.25, w_0 + 0.25 \cdot 2) = 1 + 0.5f(2.25, 1.5) = 1.78125.$$

   Then $f(t_1, w_1) = f(2.5, 1.78125) = 1.5166$. So

$$w_2 = w_1 + hf(t_1 + 0.25, w_1 + 0.25 \cdot 1.5166) = 1.78125 + 0.5f(2.75, 2.1604) = 2.4551.$$

   The errors in w_1 and w_2 are both 0.05.