Math 4650 Homework #8 Solutions

1. *Write a recursive program to use the adaptive Simpson's method as discussed in class. Set the tolerance to be* 0.00001 *and test it on the following integral:* $\int_0^2 \exp(-x^2)\, ds$. *How many times does your program have to call itself? Which points does it end up using? How far off is the answer obtained from the true value?*

   MATLAB version:

   Save this as `Simpsonizer.m`:

```
function S = Simpsonizer(f,a,b)
   S = (b-a)/6*(feval(f,a)+feval(f,b)+4*feval(f,a+(b-a)/2));
```

   Save this as `Recursive_Simpson.m`:

```
function R = Recursive_Simpson(f,a,b,tol,initial_sum)
   c = a+(b-a)/2;
   left_sum = Simpsonizer(f,a,c);
   right_sum = Simpsonizer(f,c,b);
   if abs(left_sum+right_sum-initial_sum)<=15*tol
       R = left_sum+right_sum;
   else
       R = Recursive_Simpson(f,a,c,tol/2,left_sum)
           + Recursive_Simpson(f,c,b,tol/2,right_sum);
   end
```

   Save this as `myf.m`:

```
function f = myf(x)
   f = exp(-x^2);
```

   And from the command line, run

```
Recursive_Simpson('myf', 0, 2, 0.00001, Simpsonizer('myf', 0, 2))
```

   For Mathematica:

   First command:

```
Simpsonizer[f_, a_, b_] := Module[{},
   N[(b - a)/6*(f[a] + f[b] + 4*f[a + (b - a)/2] ), 16]]
```

   Second command:

```
RecursiveSimpson[f_, a_, b_, tol_, initsum_] :=
 Module[{c, leftsum, rightsum},
  c = N[a + (b - a)/2, 16];
  leftsum := Simpsonizer[f, a, c];
  rightsum := Simpsonizer[f, c, b];
  If[Abs[leftsum + rightsum - initsum] <= 15*tol ,
   N[leftsum + rightsum, 16],
   N[RecursiveSimpson[f, a, c, tol/2, leftsum] +
     RecursiveSimpson[f, c, b, tol/2, rightsum], 16]]]
```

Third command:

```
myf[x_] := Exp[-x^2];
```

Then run command

```
RecursiveSimpson[myf, 0, 2, 0.00001, Simpsonizer[myf, 0, 2]]
```

If we take a counter, start it at zero and increment it at the first line of RecursiveSimpson, we get 13.

If we print out the points used, it ends up using the following midpoints (in order that they're needed):

$$1.000, 0.500, 0.250, 0.125, 0.375, 0.750, 0.625, 0.875, 1.500, 1.250, 1.125, 1.375, 1.750,$$

in addition to the endpoints 0.000 and 2.000.

The final answer is 0.882081633747273.

The actual answer is 0.882081390762425 to the same precision. So the error is about 2.5×10^{-7}, which is substantially smaller than the error we demanded of 10^{-5}.

2. 4.6, #9. *Let $T(a, b)$ and $T(a, (a + b)/2) + T((a + b)/2, b)$ be the single and double applications of the Trapezoidal rule to $\int_a^b f(x)\,dx$. Derive the relationship between*

$$\left| T(a, b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right|$$

and

$$\left| \int_a^b f(x)\,dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right) \right|.$$

The error in a single application of the trapezoid rule is

$$\int_a^b f(x)\,dx = T(a, b) - \frac{1}{12} f''(\xi_0) h^3,$$

by the general formula, for some $\xi \in [a, b]$.

The error in the halved interval approximation is

$$\int_a^b f(x)\,dx = T(a,c) + T(c,b) - \frac{1}{12}f''(\xi_1)\left(\frac{h}{2}\right)^3 - \frac{1}{12}f''(\xi_2)\left(\frac{h}{2}\right)^3.$$

Assuming that $\xi_0 \approx \xi_1 \approx \xi_2 = \xi$, we find

$$T(a,b) - T(a,c) - T(c,b) = -\frac{f''(\xi)}{12}h^3 + \frac{f''(\xi)}{96}h^3 + \frac{f''(\xi)}{96}h^3 = \frac{f''(\xi)}{16}h^3.$$

On the other hand

$$\int_a^b f(x)\,dx - T(a,c) - T(c,b) = -\frac{f''(\xi)}{48}h^3,$$

and from this we conclude that

$$\left|T(a,b) - T\left(a,\frac{a+b}{2}\right) - T\left(\frac{a+b}{2},b\right)\right| = 3\left|\int_a^b f(x)\,dx - T\left(a,\frac{a+b}{2}\right) - T\left(\frac{a+b}{2},b\right)\right|.$$

3. 4.6, #1a. *Compute the Simpson's rule approximations $S(a,b)$, $S(a,(a+b)/2)$, and $S((a+b)/2,b)$ for the following integrals, and verify the estimate given in the approximation formula.*

$$\int_1^{1.5} x^2 \ln x\,dx$$

We have $S(a,b) = 0.192245307413098$, $S(a,c) = 0.0393724340391205$, and $S(c,b) = 0.152886026406489$.

Therefore

$$S(a,b) - S(a,c) - S(c,b) = -0.000013153032511.$$

On the other hand,

$$\int_1^{1.5} x^2 \ln x\,dx = -\frac{19}{72} + \frac{9}{8}\ln\tfrac{3}{2} = 0.192259357732793.$$

So the error between the better approximation and the true value is

$$\int_a^b f(x)\,dx - S(a,c) - S(c,b) = 0.0000008972871835.$$

Computing, we get

$$\frac{S(a,b) - S(a,c) - S(c,b)}{\int_a^b f(x)\,dx - S(a,c) - S(c,b)} = -\frac{-0.000013153032511}{0.0000008972871835} = -14.6586653112493.$$

This is pretty close to the theoretical estimate of 15.

4. 4.7, #1a. *Approximate the following integrals using Gaussian quadrature with $n = 2$, and compare your results to the exact value of the integral.*

$$\int_1^{1.5} x^2 \ln x \, dx$$

We need to get the integral to one on $[-1, 1]$ to use Gaussian quadrature in standard form. So let $u = 4x - 5$; then $u(1) = -1$ and $u(1.5) = 1$, and we have $dx = \frac{du}{4}$, so

$$\int_1^{1.5} f(x) \, dx = \frac{1}{4} \int_{-1}^{1} f(\tfrac{u+5}{4}) \, du.$$

To approximate this latter integral using Gaussian quadrature, we use the formula

$$\int_{-1}^{1} g(u) \, du \approx g(-\tfrac{1}{\sqrt{3}}) + g(\tfrac{1}{\sqrt{3}}),$$

where $g(u) = f(\frac{u+5}{4})$. Now

$$\frac{-\frac{1}{\sqrt{3}} + 5}{4} = 1.1056624 \qquad \text{and} \qquad \frac{\frac{1}{\sqrt{3}} + 5}{4} = 1.3943376.$$

So the integral approximation is

$$\int_1^{1.5} f(x) \, dx \approx \frac{1}{4}\big(f(1.1056624)+f(1.3943376)\big) = 0.25(0.12279247+0.64628239) = 0.19226871.$$

5. *Compare your answers in the previous problems. Which does better?*

   4.6 #1 uses the composite Simpson's rule with two divisions to obtain 0.19225846.

   4.7 #1 uses Gaussian quadrature to obtain 0.19226871.

   The true value is 0.19225935.

   The composite Simpson's rule is clearly closer, although it has an advantage since it's evaluating based on *five* points in the interval, while Gaussian quadrature is only using *two* points in the interval.