Math 4650 Homework #7 Solutions

1. 4.1 #6a. *Use the most accurate three-point formula to determine each missing entry in the following tables.*

x	$f(x)$	$f'(x)$
-0.3	-0.27652	
-0.2	-0.25074	
-0.1	-0.16134	
0	0	

For $f'(-0.3)$, we use the endpoint formula (4.4) with $h = 0.1$; for $f'(-0.2)$ and $f'(-0.1)$, we use the centered difference formula (4.5); then for $f'(0)$, we again use the endpoint formula (4.4) with $h = -0.1$.

$$f'(-0.3) \approx \frac{-f(-0.1) + 4f(-0.2) - 3f(-0.3)}{0.2} = -0.060300.$$
$$f'(-0.2) \approx \frac{f(-0.1) - f(-0.3)}{0.2} = 0.57590.$$
$$f'(-0.1) \approx \frac{f(0) - f(-0.2)}{0.2} = 1.2537.$$
$$f'(0) \approx \frac{-3f(0) + 4f(-0.1) - f(-0.2)}{-0.2} = 1.9731.$$

2. 4.1 #8a. *The data in Exercise 6 were taken from the following function. Compute the actual errors in Exercise 6, and find the error bounds using the formulas.*

$$f(x) = e^{2x} - \cos 2x$$

We have $f'(x) = 2e^{2x} + 2\sin 2x$, so that the true values are

$$f'(-0.3) = -0.031661, \qquad \text{error } 0.028639$$
$$f'(-0.2) = 0.56180, \qquad \text{error } 0.014100$$
$$f'(-0.1) = 1.2401, \qquad \text{error } 0.013600$$
$$f'(0) = 2, \qquad \text{error } -0.026900$$

The error bound for formula (4.4) is $E \leq \frac{h^2}{3} f'''(\xi)$ for some $\xi \in [x, x + 2h]$, while that for (4.5) is $E \leq \frac{h^2}{6} f'''(\xi)$ for some $\xi \in [x - h, x + h]$. Obviously $f'''(x) = 8e^{2x} - 8\sin 2x$.

So

$$\begin{aligned}
f'(-0.3): \quad E &\le \left| \frac{0.1^2}{3} \max_{-0.3\le\xi\le-0.1} f'''(\xi) \right| = 0.0297 \\
f'(-0.2): \quad E &\le \left| \frac{0.1^2}{6} \max_{-0.3\le\xi\le-0.1} f'''(\xi) \right| = 0.0148 \\
f'(-0.1): \quad E &\le \left| \frac{0.1^2}{6} \max_{-0.2\le\xi\le0} f'''(\xi) \right| = 0.0141 \\
f'(0): \quad E &\le \left| \frac{0.1^2}{3} \max_{-0.2\le\xi\le0} f'''(\xi) \right| = 0.0283
\end{aligned}$$

In all cases the theoretical error is quite close to the actual error.

3. 4.2, #9. *Suppose that* $N(h)$ *is an approximation to* M *for every* $h > 0$ *and that*

$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots,$$

for some constant $K_1, K_2, K_3, \ldots$. *Use the values* $N(h)$, $H(h/3)$, *and* $N(h/9)$ *to produce an* $O(h^3)$ *approximation to* M.

We simply write the three equations and eliminate the error terms K_1 and K_2:

$$\begin{aligned}
M &= N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots \\
M &= N(\tfrac{h}{3}) + K_1 \tfrac{h}{3} + K_2 \tfrac{h^2}{9} + K_3 \tfrac{h^3}{27} + \cdots \\
M &= N(\tfrac{h}{9}) + K_1 \tfrac{h}{9} + K_2 \tfrac{h^2}{81} + K_3 \tfrac{h^3}{243} + \cdots
\end{aligned}$$

We eliminate K_1 from the first two equations to get

$$2M = 3N(\tfrac{h}{3}) - N(h) - \frac{2K_2 h^2}{3} - \frac{8K_3 h^3}{9},$$

then eliminate K_1 from the second two equations to get the highest midichlorian count I've ever seen, higher than Yoda's:

$$2M = 3N(\tfrac{h}{9}) - N(\tfrac{h}{3}) - \frac{2K_2 h^2}{27} - \frac{8K_3 h^3}{243}.$$

Finally eliminating K_2 from these two equations, we get

$$M = \frac{27N(\frac{h}{9}) - 12N(\frac{h}{3}) + N(h)}{16} + \frac{K_3 h^3}{27}.$$

4. 4.3, #16. *Let* $h = (b-a)/3$, $x_0 = a$, $x_1 = a + h$, *and* $x_2 = b$. *Find the degree of precision of the quadrature formula*

$$\int_a^b f(x)\,dx = \frac{9}{4} h f(x_1) + \frac{3}{4} h f(x_2).$$

We just need to check the formula for $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, etc. It is enough to check it for $a = 0$ and $b = 1$, since the left and right side are both invariant under scalings. So we have the formula

$$\int_0^1 f(x)\,dx = \frac{3}{4}f(\tfrac{1}{3}) + \frac{1}{4}f(1).$$

$$\begin{aligned} f(x) = 1 :& LHS = 1, RHS = \tfrac{3}{4} + \tfrac{1}{4} = 1. \\ f(x) = x :& LHS = \tfrac{1}{2}, RHS = \tfrac{3}{4}\tfrac{1}{3} + \tfrac{1}{4} = \tfrac{1}{2}. \\ f(x) = x^2 :& LHS = \tfrac{1}{3}, RHS = \tfrac{3}{4}\tfrac{1}{9} + \tfrac{1}{4} = \tfrac{1}{3}. \\ f(x) = x^3 :& LHS = \tfrac{1}{4}, RHS = \tfrac{3}{4}\tfrac{1}{27} + \tfrac{1}{4} = \tfrac{5}{18}. \end{aligned}$$

Since the formula is correct for exponents $k = 0, 1, 2$ but not for $k = 3$, the degree of precision is two.

5. 4.3, #22. *Given the function f at the following values,*

x	*1.8*	*2.0*	*2.2*	*2.4*	*2.6*
$f(x)$	*3.12014*	*4.42569*	*6.04241*	*8.03014*	*10.46675*

approximate $\int_{1.8}^{2.6} f(x)\,dx$ using all the approximate quadrature formulas of this section.

The closed Newton-Cotes formulas which make sense are $n = 1$ (trapezoid), $n = 2$ (Simpson's), and $n = 4$:

$$\begin{aligned} n = 1, h = 0.8 :& \int_{1.8}^{2.6} f(x)\,dx \approx 0.4(f(1.8) + f(2.6)) = 5.43476 \\ n = 2, h = 0.4 :& \int_{1.8}^{2.6} f(x)\,dx \approx \tfrac{0.4}{3}(f(1.8) + 4f(2.2) + f(2.6)) = 5.03420 \\ n = 4, h = 0.2 :& \int_{1.8}^{2.6} f(x)\,dx \approx \tfrac{2\cdot 0.2}{45}(7f(1.8) + 32f(2.0) + 12f(2.2) + 32f(2.4) + 7f(2.6)) = 5.03292 \end{aligned}$$

The open Newton-Cotes formulas which make sense are $n = 0$ (midpoint) and $n = 2$. (The other formulas require points which we don't have.)

$$\begin{aligned} n = 0, h = 0.4 :& \int_{1.8}^{2.6} f(x)\,dx \approx 2 \cdot 0.4 \cdot f(2.2) = 4.83393 \\ n = 2, h = 0.2 :& \int_{1.8}^{2.6} f(x)\,dx \approx \frac{4 \cdot 0.2}{3}(2f(2.0) - f(2.2) + 2f(2.4)) = 5.03180 \end{aligned}$$