

Math 4650 Homework #5 Solutions

1. 3.1 #6a. Use appropriate Lagrange interpolating polynomials of degrees one, two, and three to approximate  $f(0.43)$  if  $f(0) = 1$ ,  $f(0.25) = 1.64872$ ,  $f(0.5) = 2.71828$ ,  $f(0.75) = 4.48169$

We choose the approximating points based on proximity to the desired point:  $x_0 = 0.5$ ,  $x_1 = 0.25$ ,  $x_2 = 0.75$ ,  $x_3 = 0$ .

Using Neville's method, we get

$$0.50 \quad P_0 = 2.71828$$

$$0.25 \quad P_1 = 1.64872 \quad P_{0,1} = 4.27824x + 0.57916$$

$$0.75 \quad P_2 = 4.48169 \quad P_{1,2} = 5.66594x + 0.23223 \quad P_{0,1,2} = 5.5508x^2 + 0.11512x + 1.27302$$

$$0.00 \quad P_3 = 1.00000 \quad P_{2,3} = 4.64225x + 1.00000 \quad P_{1,2,3} = 4.09476x^2 + 1.57117x + 1.00000$$

$$P_{0,1,2,3} = 2.91208x^3 + 1.18266x^2 + 2.11721x + 1.00000$$

Then the successive approximations are  $P_{0,1}(0.43) = 2.41880$ , then  $P_{0,1,2}(0.43) = 2.34886$ , and finally  $P_{0,1,2,3}(x) = 2.36060$ .

2. 3.1 #10a. The data for Exercise 6 were generated using the following function. Use the error formula to find a bound for the error, and compare the bound to the actual error for the cases  $n = 1$  and  $n = 2$ .

$$f(x) = e^{2x}$$

The actual value is  $f(0.43) = 2.36316$ .

The general error formula is

$$f(x) - P_{0,1,\dots,n}(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x - x_0)(x - x_1) \cdots (x - x_n).$$

For  $f(x) = e^{2x}$ , we have  $f^{(n+1)}(x) = 2^{n+1}e^{2x}$  for every  $n$ .

So for  $n = 1$ , when  $x_0 = 0.50$  and  $x_1 = 0.25$ , we get

$$|f(0.43) - P_{0,1}(0.43)| = \left| \frac{4e^{2\xi}}{2!} (0.43 - 0.5)(0.43 - 0.25) \right| \leq 2e(0.07)(0.18) = 0.0685.$$

The actual error is 0.0556.

For  $n = 2$ , when  $x_0 = 0.50$  and  $x_1 = 0.25$  and  $x_2 = 0.75$ , we get

$$\begin{aligned} |f(0.43) - P_{0,1,2}(0.43)| &= \left| \frac{8e^{2\xi}}{3!} (0.43 - 0.5)(0.43 - 0.25)(0.43 - 0.75) \right| \\ &\leq \frac{4e^{1.5}}{3} (0.07)(0.18)(0.32) = 0.0241. \end{aligned}$$

The actual error is 0.0143.

3. 3.1 #14. Use the Lagrange interpolating polynomial of degree three or less and four-digit chopping arithmetic to approximate  $\cos 0.750$  using the following values. Find an error bound for the approximation.

$$\cos 0.698 = 0.7661 \quad \cos 0.733 = 0.7432 \quad \cos 0.768 = 0.7193 \quad \cos 0.803 = 0.6946$$

The actual value of  $\cos 0.750$  is 0.7317 (to four decimal places). Explain the discrepancy between the actual error and the error bound.

We use  $x_0 = 0.698$ ,  $x_1 = 0.733$ ,  $x_2 = 0.768$ , and  $x_3 = 0.803$ . Then the general formula for the Lagrange interpolating polynomial is

$$\begin{aligned} P_{0,1,2,3}(x) &= 0.7661 \frac{(x - 0.733)(x - 0.768)(x - 0.803)}{(0.698 - 0.733)(0.698 - 0.768)(0.698 - 0.803)} \\ &+ 0.7432 \frac{(x - 0.698)(x - 0.768)(x - 0.803)}{(0.733 - 0.698)(0.733 - 0.768)(0.733 - 0.803)} \\ &+ 0.7193 \frac{(x - 0.698)(x - 0.733)(x - 0.803)}{(0.768 - 0.698)(0.768 - 0.733)(0.768 - 0.803)} \\ &+ 0.6946 \frac{(x - 0.698)(x - 0.733)(x - 0.768)}{(0.803 - 0.698)(0.803 - 0.733)(0.803 - 0.768)}. \end{aligned}$$

Therefore we have, using 4-digit chopping,

$$\begin{aligned} P_{0,1,2,3}(0.750) &= \frac{(0.7661)(0.017)(-0.018)(-0.053)}{(-0.035)(-0.070)(-0.105)} + \frac{(0.7432)(0.052)(-0.018)(-0.053)}{(0.035)(-0.035)(-0.070)} \\ &+ \frac{(0.7193)(0.052)(0.017)(-0.053)}{(0.070)(0.035)(-0.035)} + \frac{(0.6946)(0.052)(0.017)(-0.018)}{(0.105)(0.070)(0.035)} \\ &= -\frac{0.00001241}{0.0002572} + \frac{0.00003686}{0.00008575} + \frac{0.00003369}{0.00008575} - \frac{0.00001104}{0.0002572} \\ &= -0.04825 + 0.4298 + 0.3928 - 0.04292 \\ &= 0.7313. \end{aligned}$$

So the actual error is  $|0.7313 - 0.7317| = 0.0004$ .

The error bound is

$$|f(x) - P_{0,1,2,3}(x)| = \left| \frac{f^{(iv)}(\xi)}{4!} (0.750 - 0.698)(0.750 - 0.733)(0.750 - 0.768)(0.750 - 0.803) \right| \leq 4 \times 10^{-8}.$$

Clearly roundoff error is responsible for the discrepancy.

4. 3.1 #20. Let  $f(x) = e^x$ , for  $0 \leq x \leq 2$ .

(a) Approximate  $f(0.25)$  using linear interpolation with  $x_0 = 0$  and  $x_1 = 0.5$ .

We have

$$f(0.25) \approx P_{0,1}(0.25) = 0.5(e^0 + e^{0.5}) = 1.324361.$$

(b) Approximate  $f(0.75)$  using linear interpolation with  $x_0 = 0.5$  and  $x_1 = 1$ .

We have

$$f(0.75) \approx P_{0,1}(0.75) = 0.5(e^{0.5} + e^1) = 2.183502.$$

(c) Approximate  $f(0.25)$  and  $f(0.75)$  by using the second interpolating polynomial with  $x_0 = 0$ ,  $x_1 = 1$ , and  $x_2 = 2$ .

The second interpolating polynomial is

$$P_{0,1,2}(x) = e^0 \frac{(x-1)(x-2)}{2} + e^1 \frac{x(x-2)}{-1} + e^2 \frac{x(x-1)}{2}.$$

So we get  $P_{0,1,2}(0.25) = 1.152774$  and  $P_{0,1,2}(0.75) = 2.011915$ .

(d) Which approximations are better and why?

The true values are  $f(0.25) = 1.284025$  and  $f(0.75) = 2.117000$ .

The errors in using  $P_{0,1}$  are 0.040336 and 0.066502.

The errors in using  $P_{0,1,2}$  are 0.131251 and 0.105085.

The linear polynomial does better for two reasons:

- The sampling points for the linear polynomial are closer to the actual point we need to estimate, so the product terms in the error formula are smaller.
- The derivative term in the error formula is also smaller, since in the second case the intervals are  $[0, 2]$ , and  $e^2$  is somewhat large compared to  $e^1$ .

## 5. Extra problem

(a) Suppose you have a function, and you know its values  $f(0)$ ,  $f(p)$ , and  $f(q)$  for three points  $x_0 = 0$ ,  $x_1 = p$ ,  $x_2 = q$ . Let  $P_{p,q}(x)$  be the second-degree interpolating polynomial for  $f$  given this data. Write a general formula for  $P_{p,q}(x)$ . Get a common denominator.

We have

$$\begin{aligned} P_{p,q}(x) &= \frac{f(0)(x-p)(x-q)}{pq} + \frac{f(p)x(x-q)}{p(p-q)} + \frac{f(q)x(x-p)}{q(q-p)} \\ &= \frac{f(0)(x-p)(x-q)(q-p) - f(p)x(x-q)q + f(q)xp(x-p)}{pq(q-p)}. \end{aligned}$$

(b) Compute the limit of  $P_{p,q}(x)$  as  $p$  approaches zero, holding  $q$  and  $x$  constant. Call this limit  $P_q(x)$ . (Hint: use L'Hopital's rule.)

As  $p$  approaches zero, we have

$$\begin{aligned} P_q(x) &= \lim_{p \rightarrow 0} P_{p,q}(x) = \lim_{p \rightarrow 0} \frac{f(0)(x-p)(x-q)(q-p) - f(p)x(x-q)q + f(q)xp(x-p)}{pq(q-p)} = \frac{0}{0} \\ &= \lim_{p \rightarrow 0} \frac{-f(0)(x-q)(q-p) - f(0)(x-p)(x-q) - f'(p)x(x-q)q + f(q)x(x-p) - f(q)xp}{q^2 - 2pq} \\ &= \frac{-f(0)q(x-q) - f(0)x(x-q) - f'(0)x(x-q)q + f(q)x^2}{q^2} \\ &= \frac{-f(0)(x^2 - q^2) - f'(0)xq(x-q) + f(q)x^2}{q^2}. \end{aligned}$$

(c) Compute the limit of  $P_q(x)$  as  $q$  approaches zero, holding  $x$  constant. Show that you get the second Maclaurin polynomial.

$$\begin{aligned} P(x) &= \lim_{q \rightarrow 0} P_q(x) = \lim_{q \rightarrow 0} \frac{-f(0)(x^2 - q^2) - f'(0)xq(x - q) + f(q)x^2}{q^2} = \frac{0}{0} \\ &= \lim_{q \rightarrow 0} \frac{2f(0)q - f'(0)x(x - q) + f'(0)xq + f'(q)x^2}{2q} = \frac{0}{0} \\ &= \lim_{q \rightarrow 0} \frac{2f(0) + f'(0)x + f'(0)x + f''(q)x^2}{2} \\ &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2, \end{aligned}$$

which is the second Maclaurin polynomial for  $f$ .