

Math 4650 Homework #4 Solutions

1. Suppose p_n is a sequence which is known to converge **quadratically** to p . Imitate the technique at the beginning of Section 2.5 (or from class) to find a sequence that would be expected to converge faster.

Solution: Since the sequence converges quadratically, we expect

$$\frac{p_{n+2} - p}{(p_{n+1} - p)^2} \approx \frac{p_{n+1} - p}{(p_n - p)^2}.$$

Assuming this approximation is exact, we obtain the equation

$$(p_{n+2} - p)(p_n - p)^2 = (p_{n+1} - p)^3.$$

Expanding out, we get

$$p_{n+2}p_n^2 - 2p_{n+2}p_np + p_{n+2}p^2 - p_n^2p + 2p_np^2 - p^3 = p_{n+1}^3 - 3p_{n+1}^2p + 3p_{n+1}p^2 - p^3.$$

The p^3 terms cancel out, and bringing everything over to the left side, we get

$$(2p_n + p_{n+2} - 3p_{n+1})p^2 + (3p_{n+1}^2 - p_n^2 - 2p_{n+2}p_n)p + p_{n+2}p_n^2 - p_{n+1}^3 = 0.$$

This is a quadratic equation for p , which we can solve to get

$$p = \frac{-(3p_{n+1}^2 - p_n^2 - 2p_{n+2}p_n) \pm \sqrt{(3p_{n+1}^2 - p_n^2 - 2p_{n+2}p_n)^2 - 4(2p_n + p_{n+2} - 3p_{n+1})(p_{n+2}p_n^2 - p_{n+1}^3)}}{2(2p_n + p_{n+2} - 3p_{n+1})}.$$

The only question is whether to choose the plus sign or the minus sign. One way to find out is to take a typical term of quadratically-convergent sequence, of the form

$$p_n = p + \frac{1}{\lambda} r^{2^n}.$$

Then we have exactly

$$\frac{p_{n+1} - p}{(p_n - p)^2} = \frac{\frac{r^{2^{n+1}}}{\lambda}}{\left(\frac{r^{2^n}}{\lambda}\right)^2} = \lambda.$$

Writing $R = r^{2^n}$, we see that $p_n = p + R/\lambda$, $p_{n+1} = p + R^2/\lambda$, and $p_{n+2} = p + R^4/\lambda$. We then find that the quadratic formula is

$$\hat{p} = \frac{(-6p\lambda R^2 - 3R^4 + 4p\lambda R + R^2 + 2p\lambda R^4 + 2R^5) \pm R^2(2R + 1)(R - 1)^2}{2\lambda R(R + 2)(R - 1)^2}.$$

If we choose the plus sign, this becomes

$$\hat{p} = \frac{2R^2 + R + p\lambda R + 2p\lambda}{\lambda(R + 2)}.$$

If we choose the minus sign, it becomes

$$\hat{p} = p.$$

Obviously, therefore, we want the minus sign, so that the accelerated sequence should in general be

$$\hat{p}_n = \frac{-(3p_{n+1}^2 - p_n^2 - 2p_{n+2}p_n) - \sqrt{(3p_{n+1}^2 - p_n^2 - 2p_{n+2}p_n)^2 - 4(2p_n + p_{n+2} - 3p_{n+1})(p_{n+2}p_n^2 - p_{n+1}^3)}}{2(2p_n + p_{n+2} - 3p_{n+1})}.$$

We also have to worry about roundoff error resulting from subtracting two nearly-equal numbers, but this is getting ridiculous. You can see why it's generally not practically worthwhile to accelerate quadratically-convergent sequences.

2. Verify that the iterative method given in Section 2.4, problem 13 (Newton's method of order 3) actually has $g'(p) = 0$ and $g''(p) = 0$. Conclude that it is a method of order 3. Find the coefficient $g'''(p)$ in terms of the values of f and its derivatives at p . Use this to get the asymptotic error constant λ .

Solution: The function appearing in the third-order Newton's method is

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f(x)^2 f''(x)}{2f'(x)^3}.$$

We just have to compute the derivatives. Using Maple, we have

$$\begin{aligned} g'(x) &= f(x)^2 \frac{3f''(x)^2 - f'''(x)f'(x)}{2f'(x)^4}, \\ g''(x) &= -\frac{f(x)}{2f'(x)^5} \left(f^{(iv)}(x)f(x)f'(x)^2 + 2f'''(x)f'(x)^3 - 9f'''(x)f(x)f''(x)f'(x) \right. \\ &\quad \left. - 6f''(x)^2 f'(x)^2 + 12f''(x)^3 f(x) \right), \\ g'''(x) &= -\frac{1}{2f'(x)^6} \left(f^{(v)}(x)f(x)^2 f'(x)^3 + 4f^{(iv)}(x)f(x)f'(x)^4 - 12f^{(iv)}(x)f(x)^2 f''(x)f'(x)^2 \right. \\ &\quad \left. + 2f'''(x)f'(x)^5 - 34f'''(x)f(x)f''(x)f'(x)^3 + 72f'''(x)f(x)^2 f''(x)^2 f'(x) \right. \\ &\quad \left. - 9f'''(x)^2 f(x)^2 f'(x)^2 - 6f''(x)^2 f'(x)^4 + 42f''(x)^3 f(x)f'(x)^2 - 60f''(x)^4 f(x)^2 \right). \end{aligned}$$

And then you just plug in $x = p$ and use, many times, the fact that $f(p) = 0$.

(If you were going to do this by hand, you'd try to look for $f(p)$ terms to pull out long before you went through all that trouble of computing every single term.)

Assuming that $f'(p) \neq 0$ (otherwise everything breaks down), we get $g'(p) = 0$, $g''(p) = 0$, and

$$g'''(p) = 3 \frac{f''(p)^2}{f'(p)^2} - \frac{f'''(p)}{f'(p)}.$$

So as long as $f'(p) \neq 0$ and $f'''(p)f'(p) - 3f''(p)^2 \neq 0$, we get $g'''(p) \neq 0$.

Then finally we have

$$p_n - p \approx \frac{g'''(p)}{6} (p_n - p)^3,$$

which means (just reading off the equation) that $\alpha = 3$ and that

$$\lambda = \frac{|g'''(p)|}{6} = \frac{|3f''(p)^2 - f'''(p)f'(p)|}{6f'(p)^2}.$$