

Math 4650 Homework #4 Solutions

- 2.2 #7. Use Theorem 2.2 to show that  $g(x) = \pi + 0.5 \sin(x/2)$  has a unique fixed point on  $[0, 2\pi]$ . Use fixed-point iteration to find an approximation to the fixed point that is accurate to within  $10^{-2}$ . Use Corollary 2.4 to estimate the number of iterations required to achieve  $10^{-2}$  accuracy, and compare this theoretical estimate to the number actually needed.

Since  $-1 \leq \sin \leq 1$ , we see that  $\pi - 0.5 \leq g(x) \leq \pi + 0.5$  for every  $x$ ; in particular  $0 \leq g(x) \leq 2\pi$  for every  $x$ . So  $g$  satisfies condition (a) of Theorem 2.2. Furthermore we have  $g'(x) = 0.25 \cos(x/2)$ , so that  $|g'(x)| \leq 0.25$  for every  $x$ . Thus condition (b) is satisfied as well, using  $k = \frac{1}{4}$ . So there is a unique fixed point.

For the actual approximation, we start at the midpoint of  $[0, 2\pi]$ , with  $p_0 = \pi$ . We get the sequence

$$\begin{aligned} p_0 &= 3.141592654 \\ p_1 &= 3.641592654 \\ p_2 &= 3.626048865 \\ p_3 &= 3.626995623 \\ p_4 &= 3.626938795 \\ p_5 &= 3.626942209 \\ p_6 &= 3.626942004 \end{aligned}$$

which rounds off to  $p \approx 3.63$ , and this accuracy is attained already at  $p_2$ .

One formula in Corollary 2.4 says that the error would theoretically be at most

$$|p_n - p| \leq \frac{1}{4^n} \max\{p_0 - a, b - p_0\} = \frac{\pi}{4^n}.$$

So theoretically if we wanted the error guaranteed to at most 0.01, we'd choose  $n$  so that

$$\frac{\pi}{4^n} \leq 0.01, \quad \text{i.e., } n > 4.$$

The other formula in Corollary 2.4 tells us to use  $p_1 - p_0$ , and this formula gives

$$|p_n - p| \leq \frac{4}{3} \frac{1}{4^n} (0.5) = 0.01,$$

which gives  $n > 3$ .

Either way,  $n = 2$  is sufficient so the theoretical prediction is pessimistic (as usual).

- 2.3 #1. Let  $f(x) = x^2 - 6$  and  $p_0 = 1$ . Use Newton's method to find  $p_2$ .

We have

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 6}{2x}.$$

Hence  $p_1 = g(1) = 3.5$ , and  $p_2 = g(3.5) = 2.607143$ .

The actual square root of 6 is 2.449490.

- 2.3 #3. Let  $f(x) = x^2 - 6$ . With  $p_0 = 3$  and  $p_1 = 2$ , find  $p_3$ .

a. Use the Secant method.

We have

$$p_n = p_{n-1} - f(p_{n-1}) \frac{p_{n-1} - p_{n-2}}{f(p_{n-1}) - f(p_{n-2})},$$

so that  $p_2 = 2.4000$  and  $p_3 = 2.454545$ .

b. Use the method of False Position.

We have to verify the signs:  $f(2) = -2 < 0$  and  $f(3) = 3 > 0$ , so there is a root. Finding the test point is the same as the secant method:  $p_2 = 2.4000$ .

Now we find the new interval:  $f(2.4) = -0.24$ , so that our new interval is  $[2.4, 3.0]$ . Using the secant method with these two points, we get  $p_3 = 2.444444$ .

c. Which of (a) or (b) is closer to  $\sqrt{6}$ ?

The secant method has error 0.005055, while the false position method has error 0.005045. They are nearly indistinguishable, but false position happens to be slightly closer this time.

- 2.3 #12. Use all three methods in this section to find solutions to within  $10^{-7}$  for the following problems.

a.  $x^2 - 4x + 4 - \ln x = 0$  for  $1 \leq x \leq 2$  and for  $2 \leq x \leq 4$

Newton For  $1 \leq x \leq 2$ : using  $p_0 = 1.5$ , we get

$$p_0 = 1.5000000$$

$$p_1 = 1.4067209$$

$$p_2 = 1.4123700$$

$$p_3 = 1.4123912$$

and all further outputs are the same.

For  $2 \leq x \leq 4$ : using  $p_0 = 3$ , we get

$$p_0 = 3.0000000$$

$$p_1 = 3.0591674$$

$$p_2 = 3.0571061$$

$$p_3 = 3.0571035$$

and all further outputs are the same.

Secant For  $1 \leq x \leq 2$ , using  $p_0 = 1$  and  $p_1 = 2$ , we get

$$\begin{aligned}p_0 &= 1.0000000 \\p_1 &= 2.0000000 \\p_2 &= 1.5906161 \\p_3 &= 1.2845478 \\p_4 &= 1.4279661 \\p_5 &= 1.4136346 \\p_6 &= 1.4123782 \\p_7 &= 1.4123912\end{aligned}$$

For  $2 \leq x \leq 4$ , using  $p_0 = 2$  and  $p_1 = 4$ , we get

$$\begin{aligned}p_0 &= 2.0000000 \\p_1 &= 4.0000000 \\p_2 &= 2.4192186 \\p_3 &= 2.7560397 \\p_4 &= 3.3170225 \\p_5 &= 3.0097690 \\p_6 &= 3.0506707 \\p_7 &= 3.0572890 \\p_8 &= 3.0571028 \\p_9 &= 3.0571035\end{aligned}$$

False p For  $1 \leq x \leq 2$ , using  $a = 1$  and  $b = 2$  as the initial endpoints, the resulting endpoints are

$$\begin{aligned}&[1.0000000, 2.0000000] \\&[1.0000000, 1.5906161] \\&[1.0000000, 1.4555373] \\&[1.0000000, 1.4222100] \\&[1.0000000, 1.4145936] \\&[1.0000000, 1.4128836] \\&[1.0000000, 1.4125012] \\&[1.0000000, 1.4124157] \\&[1.0000000, 1.4123967] \\&[1.0000000, 1.4123924] \\&[1.0000000, 1.4123914] \\&[1.0000000, 1.4123912]\end{aligned}$$

For  $2 \leq x \leq 4$ , using  $a = 2$  and  $b = 4$  as the endpoints, we get

[2.4192186, 4.0000000]  
 [2.7560397, 4.0000000]  
 [2.9360446, 4.0000000]  
 [3.0119816, 4.0000000]  
 [3.0407874, 4.0000000]  
 [3.0512697, 4.0000000]  
 [3.0550261, 4.0000000]  
 [3.0563648, 4.0000000]  
 [3.0568410, 4.0000000]  
 [3.0570103, 4.0000000]  
 [3.0570704, 4.0000000]  
 [3.0570918, 4.0000000]  
 [3.0570994, 4.0000000]  
 [3.0571021, 4.0000000]  
 [3.0571030, 4.0000000]  
 [3.0571034, 4.0000000]  
 [3.0571035, 4.0000000]

b.  $x + 1 - 2 \sin \pi x = 0$  for  $0 \leq x \leq \frac{1}{2}$  and for  $\frac{1}{2} \leq x \leq 1$

Here I'll just present the answers (0.2060351 and 0.6819748) and how long it takes.

Newton's method on  $[0, \frac{1}{2}]$  reaches the answer in 3 steps. On  $[\frac{1}{2}, 1]$  it also reaches in 3 steps.

The secant method on  $[0, \frac{1}{2}]$  reaches the answer in 7 steps. On  $[\frac{1}{2}, 1]$  it reaches in 6 steps.

False position on  $[0, \frac{1}{2}]$  reaches the answer to within  $10^{-7}$  in 11 steps. On  $[\frac{1}{2}, 1]$  it reaches the answer in 13 steps.

- 2.3 #19. *The iteration equation for the Secant method can be written in the simpler form*

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

*Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.*

Both formulas involve subtracting two nearly equal numbers in both the numerator and denominator, which results in a large loss of precision. We then divide these two low-precision numbers and get a result with very low precision. However, in the actual algorithm we use this low-precision number only to correct an estimate we already have (so it's OK if we only know the correction term to one digit, for example, since it's only correcting the last digit of the previous value). In the formula above, our new

approximation is the low-precision value, so we have lost a bunch of decimal places of precision by doing this.

- 2.4 #6. Show that the following sequences converge linearly to  $p = 0$ . How large must  $n$  be before we have  $|p_n - p| \leq 5 \times 10^{-2}$ ?

a.  $p_n = \frac{1}{n}, \quad n \geq 1.$

We have

$$\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1,$$

so we have  $\alpha = 1$  and  $\lambda = 1$ . We must have  $n \geq 20$  to be within the desired error.

b.  $p_n = \frac{1}{n^2}, \quad n \geq 1.$

We have

$$\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = 1,$$

so  $\alpha = 1$  and  $\lambda = 1$  again. We must have  $n \geq 5$  to be within the desired error.

- 2.4 #9.

a. Construct a sequence that converges to 0 of order 3.

b. Suppose  $\alpha > 1$ . Construct a sequence that converges to 0 of order  $\alpha$ .

The answer for both is about the same: just choose

$$p_n = \frac{1}{2^{\alpha^n}}$$

and we get

$$\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n^\alpha} = \lim_{n \rightarrow \infty} \frac{(2^{\alpha^n})^\alpha}{2^{\alpha^{n+1}}} = \lim_{n \rightarrow \infty} 1 = 1.$$

So  $\lambda = 1$ , and hence the order is actually  $\alpha$ .

- 2.5 #4. Let  $g(x) = 1 + (\sin x)^2$  and  $p_0^{(0)} = 1$ . Use Steffensen's method to find  $p_0^{(1)}$  and  $p_0^{(2)}$ .

The new function is

$$h(x) = x - \frac{[g(x) - x]^2}{g(g(x)) - 2g(x) + x},$$

and we have

$$p_0^{(n+1)} = h(p_0^{(n)}).$$

Therefore  $p_0^{(1)} = 2.152904628$  and  $p_0^{(2)} = 1.873464044$ .