Math 4650 Homework #3 Solutions

- 1.3 #7. Find the rates of convergence of the following functions as $h \to 0$.

1. $\lim_{h\to 0} \frac{\sin h}{h} = 1.$
   Use $\sin h = h - \frac{h^3}{6} + \cdots$, and we get
   $$\frac{\sin h}{h} - 1 = \frac{\sin h - h}{h} = -\frac{h^3}{6h} + \cdots = -\frac{h^2}{6} + \cdots .$$
   So it's $O(h^2)$.
2. $\lim_{h\to 0} \frac{1-\cos h}{h} = 0.$
   Use $\cos h = 1 - \frac{h^2}{2} + \cdots$ to get
   $$\frac{1-\cos h}{h} = \frac{1 - \left(1 - \frac{h^2}{2} + \cdots\right)}{h} = \frac{h}{2} + \cdots .$$
   So the order is $O(h)$.
3. $\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0.$
   Use $\sin h = h - \frac{h^3}{6} + \cdots$ and $\cos h = 1 - \frac{h^2}{2} + \cdots$, and we find
   $$\sin h - h\cos h = h - \frac{h^3}{6} - h + \frac{h^3}{2} + \cdots = \frac{h^3}{3} + \cdots .$$
   Therefore
   $$\frac{\sin h - h\cos h}{h} = \frac{h^2}{3} + \cdots ,$$
   and the order is $O(h^2)$.
4. $\lim_{h\to 0} \frac{1-e^h}{h} = -1.$
   Use $e^h = 1 + h + \frac{h^2}{2} + \cdots$, and get
   $$\frac{1-e^h}{h} + 1 = \frac{1+h-e^h}{h} = \frac{1 + h - 1 - h - \frac{h^2}{2} - \cdots}{h} = -\frac{h}{2} + \cdots$$
   so that the order is $O(h)$.

- 1.3, #8.

1. How many multiplications and additions are required to determine a sum of the form
   $$\sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j ?$$

Solution: We clearly have one multiplication for each summand, so there are $\sum_{i=1}^{n}\sum_{j=1}^{i}(1) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ multiplications. We then have $(i-1)$ additions for the internal sum, as i goes from 1 to n, giving a total of $\frac{n(n-1)}{2}$ additions internally. Then another $(n-1)$ additions to do the outer sum, so we get $\frac{n(n+1)}{2}-1$ additions in total.

2. Modify the sum in part (a) to an equivalent form that reduces the number of computations.

   Solution: We have

$$\sum_{i=1}^{n}\sum_{j=1}^{i} a_i b_j = \sum_{i=1}^{n} a_i \sum_{j=1}^{i} b_j.$$

   This has the same number of additions, but we have reduced the multiplications to only n.

   We can reduce this further: observe that if we save the numbers $c_i = \sum_{j=1}^{i} b_j$, then having each c_i means that getting c_{i+1} is only one more addition. So there are really only $(n-1)$ additions to determine c_1 through c_n. Then we have n multiplications and $(n-1)$ additions, for a total of n multiplications and $2(n-1)$ additions.

- 2.1, #4. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on each interval.

  Solution: Using the Mathematica bisection program given in class with a tolerance of 0.01, we obtain the following results:

  1. $[-2,-1] : -1.41406$
  2. $[0,2] : 1.41406$
  3. $[2,3] : 2.72656$
  4. $[-1,0] : -0.726563$

- 2.1, #10. Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the following intervals?

  Solution: Using the Mathematica bisection program, we obtain the following answers.

  1. $[-1.5, 2.5] : 0$
  2. $[-0.5, 2.4] : 0$
  3. $[-0.5, 3] : 2$
  4. $[-3, -0.5] : -2$

- 2.2, #4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

  1. $p_n = p_{n-1}\left(1 + \frac{7 - p_{n-1}^5}{p_{n-1}^2}\right)^3$

2. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$
3. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$
4. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$

The Mathematica code looks something like this:

```
x = 1;
Maxiter = 20;
Counter = 1;
Tol = 0.001;
Err = Tol + 1;
While[((Err > Tol) && (Counter <= Maxiter)),
 {x = N[x*(1 + (7 - x^5)/x^2)^3];
  Err = Abs[x^5 - 7];
  Counter += 1;
  Print[x];}]
If[Err > Tol, Print["Failed to reach tolerance"],
 Print["Reached tolerance"]]
```

In case (a) we get the sequence

$$1, 343, -2 \times 10^{25}, -3 \times 10^{253}, \cdots$$

which very quickly leads to overflow error.

In case (b) we get the sequence

$$1, 7, -336, 4 \times 10^7, -5 \times 10^{22}, 2 \times 10^{68}, \cdots$$

which takes only a little longer to reach overflow error.

In case (c) we get the sequence

$$1, 2.2, 1.81976, 1.58347, 1.48946, 1.47602, 1.47577, 1.47577$$

which seems correct to within 10^{-6}.

In case (d) we get the sequence

$$1, 1.5, 1.451, 1.499, 1.452, 1.498, 1.453, 1.496, 1.454, 1.495, \cdots$$

It is hard to tell whether this sequence is actually converging; it looks like it may just be going back and forth around the root without actually getting very close to the root.

We can explain these results easily using the general fixed point theorem. In case (a) we have $g'(p) = -47$, where $p = 7^{1/5}$ is the fixed point. In case (b) we have $g'(p) = -10$. Clearly also divergent but not as fast as case (a). In case (c) we have $g'(p) = 0$, so convergent quite quickly. In case (d) we have $g'(p) = -0.98$, so that the iteration is just barely convergent.