

Math 4650 Homework #2 Solutions

- 1.2 #3ab. Suppose  $p^*$  must approximate  $p$  with relative error at most  $10^{-3}$ . Find the largest interval in which  $p^*$  must lie for each value of  $p$ .

**Solution:** The relative error is

$$\frac{|p^* - p|}{p} = \varepsilon,$$

so that we must have

$$\begin{aligned} (1 - \varepsilon)p &\leq p^* \leq (1 + \varepsilon)p \\ 0.999p &\leq p^* \leq 1.001p. \end{aligned}$$

1. For  $p = 150$  we have  $149.85 \leq p^* \leq 150.15$ .
  2. For  $p = 900$  we have  $899.1 \leq p^* \leq 900.9$ .
- 1.2 #5beh. Use three-digit rounding arithmetic to perform the following calculations. Compute the absolute error and relative error with the exact value determined to at least five digits.

**Solution:**

- b.  $133 - 0.499$ . The exact answer to five digits is 132.50, which rounded to three digits is 133.

- e.  $\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$ . The exact answer is 1.9535 to five digits.

Using rounding arithmetic, we get  $\frac{13}{14} = 0.929$  and  $\frac{6}{7} = 0.857$  so that  $\frac{13}{14} - \frac{6}{7} = 0.0720$ . We have  $e = 2.72$  so that  $2e = 5.44$ , and  $2e - 5.4 = 0.0400$ . Hence

$$\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4} = \frac{0.0720}{0.0400} = 1.80.$$

- h.  $\frac{\pi - \frac{22}{7}}{\frac{1}{17}}$ . The exact answer is  $-0.021496$  to five digits.

Using rounding arithmetic, we have  $\pi = 3.14$  and  $\frac{22}{7} = 3.14$ , so that  $\pi - \frac{22}{7} = 0.00$ . It happens that  $\frac{1}{17} = 0.0588$ , though this hardly matters since

$$\frac{\pi - \frac{22}{7}}{\frac{1}{17}} = \frac{0.00}{0.0588} = 0.00.$$

- 1.2 #12. Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

1. Find  $\lim_{x \rightarrow 0} (e^x - e^{-x})/x$ .

**Solution:** Using L'Hopital's rule, we get

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{1} = 2.$$

2. Use three-digit rounding arithmetic to evaluate  $f(0.1)$ .

**Solution:** We have  $e^{0.1} = 1.11$  and  $e^{-0.1} = 0.905$ , so that  $e^{0.1} - e^{-0.1} = 0.205$ . Therefore  $\frac{e^{0.1} - e^{-0.1}}{0.1} = 2.05$ , to three digits.

3. Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).

**Solution:** We have  $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$  and  $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$ , so that

$$\frac{e^x - e^{-x}}{x} = \frac{2x + \frac{x^3}{3}}{x} = 2 + \frac{x^2}{3}.$$

Thus we can compute

$$f(0.1) \approx 2 + \frac{0.1^2}{3} = 2.0033$$

which rounds to 2.00.

4. The actual value is  $f(0.1) = 2.003335000$ . Find the relative error for the values obtained in parts (b) and (c).

**Solution:** For part (b) the relative error is 2.3%. For part (c) the relative error is 0.17%.

Extra. Suppose you want to compute  $e$  numerically. Using the Maclaurin series, we know

$$e = 1 + 1/1 + 1/(1 * 2) + 1/(1 * 2 * 3) + 1/(1 * 2 * 3 * 4) + \dots$$

1. Using Taylor's theorem, estimate the error in using  $n$  terms of the Maclaurin series for  $e^x$  at  $x = 1$ .

**Solution:** The remainder term is

$$R_n(1) = \frac{f^{(n+1)}(\xi)1^{n+1}}{(n+1)!} = \frac{e^\xi}{(n+1)!} \leq \frac{3}{(n+1)!}.$$

2. Show that you can get the correct value to within 0.03 by using the terms up to  $n = 4$ .

**Solution:** When  $n = 4$  we get  $R_n(1) \leq \frac{3}{5!} = 0.025 < 0.03$ .

3. Write some pseudocode to compute  $e$  using this approximation.

**Solution:**

```
E = 1
FACTORIAL = 1
FOR K FROM 1 TO N DO
    FACTORIAL = FACTORIAL * K
    E = E + 1/FACTORIAL
END DO
PRINT E
```

4. Which do you expect to be more efficient? Why?

$$1 + 1/1 + 1/(1 * 2) + 1/(1 * 2 * 3) + 1/(1 * 2 * 3 * 4)$$

or

$$1 + 1/1 + 1/2 * (1 + 1/3 * (1 + 1/4))$$

**Solution:** The latter method involves fewer computations, and therefore it will probably be both faster and have less roundoff error.

5. Try using both methods on a calculator. Round to three significant digits after every computation. Which does better?

**Solution:** Using the first method, we have the sequence

$$1.00, 2.00, 2.50, 2.67, 2.71.$$

Using the second method, we have the sequence

$$1.25, 1.42, 1.71, 2.71.$$