

Math 4650 Homework #2 Solutions

1. The continued fraction expansion of the "golden ratio" $\phi = (1 + \sqrt{5})/2$ is

$$1 + 1/(1 + 1/(1 + 1/(1 + 1/(1 + \dots))))).$$

So for example we can approximate ϕ with the three-term continued fraction $1 + 1/(1 + 1) = 1.5$ or with the four-term continued fraction $1 + 1/(1 + 1/(1 + 1)) = 1 + 1/(3/2) = 1.67$.

Write a program with a FOR loop to approximate ϕ with the N -term continued fraction. Test your program with $N = 3$ and $N = 4$ and make sure you get the answers above. Plug in various values of N and see how many terms you need to get within 0.0001 of the true value.

Solution: There are a few ways to do this. One is to start with 2, take the reciprocal, add one, take the reciprocal, add one, and repeat. These two commands (taking the reciprocal and adding one) are repeated $N - 2$ times. So a quick pseudocode algorithm would be

```
Phi = 2
For j from 1 to N-2 do
    Phi = 1+1/Phi
End
Print(Phi)
```

When $N = 3$, this reproduces $\phi = \frac{3}{2}$, and when $N = 4$, this gives $\phi = \frac{5}{3}$.

The true value of ϕ is 1.618033988 to ten digits. If we use $N = 11$, we obtain an estimate of 1.617977528, which is less than 0.0001 from the true answer.

2. Recall the difficulty with using the quadratic formula computationally, due to the possibility of subtracting two numbers that are nearly equal.

Write a program that takes as input the numbers a , b , and c , the coefficients of a quadratic equation $ax^2 + bx + c = 0$. The program should compute the real solutions of this equation. (If there are none, the program should say so.) Use the proper form of the quadratic formula (either equation 1.1, 1.2, or 1.3 in Section 1.2) to compute each root in such a way that you never subtract two numbers that have the same sign. (You'll need to use several IF...THEN...ELSE statements but no loops.) Check your program by running it on the following equations, using 4-digit rounding: $2.1x^2 - 42.6x + 1.0 = 0$ and $2.0x^2 + 55x - 1.0 = 0$. Make sure you get answers correct to all four places.

Solution: The first root is

$$\begin{cases} \frac{-b + \sqrt{b^2 - 4ac}}{2a} & b < 0, \\ \frac{-2c}{b + \sqrt{b^2 - 4ac}} & b > 0, \end{cases}$$

while the second root is

$$\begin{cases} \frac{-2c}{b - \sqrt{b^2 - 4ac}} & b < 0, \\ \frac{b + \sqrt{b^2 - 4ac}}{2a} & b > 0. \end{cases}$$

A pseudocode would look like

```

# A program to solve  $ax^2 + bx + c = 0$ .
a = 2.1
b = -42.6
c = 1.0
# First check degeneracy: is the equation linear or constant?
If (a=0) Then
  If (b=0) Then
    Print "Constant function: no roots."
  Else
    x = -c/b
    Print "Linear function: one root at ", x
  End If
Else
  Disc =  $b^2 - 4*a*c$ 
  If (Disc<0) Then
    Print "Negative discriminant: no real roots."
  Else
    If (Disc=0) Then
      x = -b/(2*a)
      Print "Zero discriminant: one repeated root at ", x
    Else
      # We have negative discriminant; check sign of b.
      sqd = sqrt(Disc)
      If (b<0) Then
        x1 = (-b + sqd)/(2*a)
        x2 = (-2*c)/(b - sqd)
      Else
        If (b>0) Then
          x1 = (-b - sqd)/(2*a)
          x2 = (-2*c)/(b + sqd)
        Else
          # Degenerate case, when b=0
          x1 = sqrt(-c/a)
          x2 = -x1
        End If
      End If
      Print "Two real roots at ", x1, " and ", x2
    End If
  End If
End If

```

With four-digit arithmetic, this program does indeed produce the correct values for the two given equations: $2.1x^2 - 42.6x + 1.0 = 0$ and $2.0x^2 + 55x - 1.0 = 0$. (See Maple worksheet.)

3. Textbook, Section 1.3: 1a.

Use three-digit chopping arithmetic to compute the sum $\sum_{i=1}^{10} (1/i^2)$ first by $\frac{1}{1} + \frac{1}{4} + \dots + \frac{1}{100}$ and then by $\frac{1}{100} + \frac{1}{81} + \dots + \frac{1}{1}$. Which method is more accurate, and why?

Solution: The ten numbers we're adding up are, to three-digit chopping,

True	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$	$\frac{1}{36}$	$\frac{1}{49}$	$\frac{1}{64}$	$\frac{1}{81}$	$\frac{1}{100}$
Chop	1.00	0.250	0.111	0.0625	0.0400	0.0277	0.0204	0.0156	0.0123	0.0100

If we add these from left to right and chop after every addition, we obtain the sequence

$$1.00, 1.25, 1.36, 1.42, 1.46, 1.48, 1.50, 1.51, 1.52, 1.53.$$

If we add these numbers from right to left and chop after every addition, we obtain the sequence

$$0.0100, 0.0223, 0.0379, 0.0583, 0.0860, 0.126, 0.188, 0.299, 0.549, 1.54.$$

The exact sum to ten places is 1.549767731. The second method is more accurate. The reason is that when we start with a large number and add a small number (e.g., adding 1.50 to 0.0156), all the information in the second number is lost except for the first digit. However if we add two small numbers together (e.g. 0.0223 to 0.0156) all the information is preserved.

The little bits that get chopped off in each computation will eventually add up to something substantial (much like in the plot of Superman III or Office Space), and one is much more likely to notice them and allow them to accumulate if one starts computing with the small numbers.

4. Textbook, Section 1.3: 7.

Find the rates of convergence of the following functions as $h \rightarrow 0$.

(a) $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$

Solution: We use the Maclaurin series for \sin , which is $\sin h = h - \frac{h^3}{6} + \frac{h^5}{120} + \dots$. Then we have

$$\begin{aligned} \frac{\sin h}{h} - 1 &= \frac{\sin h - h}{h} \\ &= \frac{\left(h - \frac{h^3}{6} + \frac{h^5}{120} + \dots\right) - h}{h} \\ &= -\frac{h^2}{6} + \frac{h^4}{120} + \dots \\ &= O(h^2). \end{aligned}$$

$$(b) \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0.$$

Solution: The Maclaurin series for \cos is $\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{24} + \dots$. We have

$$\begin{aligned} \frac{1 - \cos h}{h} &= \frac{1 - \left(1 - \frac{h^2}{2} + \frac{h^4}{24} + \dots\right)}{h} \\ &= \frac{\frac{h^2}{2} - \frac{h^4}{24} + \dots}{h} \\ &= \frac{h}{2} - \frac{h^3}{24} + \dots \\ &= O(h). \end{aligned}$$

$$(c) \lim_{h \rightarrow 0} \frac{\sin h - h \cos h}{h} = 0.$$

Solution: We combine the series for $\sin h$ and $\cos h$.

$$\begin{aligned} \frac{\sin h - h \cos h}{h} &= \frac{\left(h - \frac{h^3}{6} + \frac{h^5}{120} + \dots\right) - h \left(1 - \frac{h^2}{2} + \frac{h^4}{24} + \dots\right)}{h} \\ &= \frac{h - h - \frac{h^3}{6} + \frac{h^3}{2} + \frac{h^5}{120} - \frac{h^5}{24} + \dots}{h} \\ &= \frac{\frac{h^3}{3} - \frac{h^5}{30} + \dots}{h} \\ &= \frac{h^2}{3} - \frac{h^4}{30} + \dots \\ &= O(h^2). \end{aligned}$$

$$(d) \lim_{h \rightarrow 0} \frac{1 - e^h}{h} = -1.$$

Solution: We use the series for the exponential, $e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots$. Then

$$\begin{aligned} \frac{1 - e^h}{h} + 1 &= \frac{1 - e^h + h}{h} \\ &= \frac{1 + h - \left(1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots\right)}{h} \\ &= \frac{-\frac{h^2}{2} - \frac{h^3}{6} + \dots}{h} \\ &= -\frac{h}{2} - \frac{h^2}{6} + \dots \\ &= O(h). \end{aligned}$$