Math 4650 Homework #1 Solutions

1. 1.1 #2. Find intervals containing solutions to the following equations.

   Solution: For all of these problems, we need to use the intermediate value theorem. So we want to find numbers a and b such that $f(a) > 0$ and $f(b) < 0$; then there is a solution to $f(x) = 0$ for some x between a and b.

   (a) $x - 3^{-x} = 0$. The function is $f(x) = x - 3^{-x}$. Use trial and error: $f(0) = -1$, $f(1) = 1 - \frac{1}{3} = \frac{2}{3}$. Since $f(0)$ and $f(1)$ have opposite signs, there is a solution in the interval $(0, 1)$.

   (b) $4x^2 - e^x = 0$. The function is $f(x) = 4x^2 - e^x$. $f(0) = -1$, $f(1) = 4 - e = 1.3\ldots$, so again there is a solution in the interval $(0, 1)$.

   (c) $x^3 - 2x^2 - 4x + 3 = 0$. The function is $f(x) = x^3 - 2x^2 - 4x + 3$. We have $f(0) = 3$, $f(1) = -2$, so again there is a solution in $(0, 1)$.

   (d) $x^3 + 4.001x^2 + 4.002x + 1.101 = 0$. We have $f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$, so that $f(0) = 1.101$, $f(1) = 10.104$, $f(2) > 33$, etc. We're going in the wrong direction. In the other direction, we have $f(-1) = -1 + 4.001 - 4.002 + 1.101 = 0.1$, $f(-2) = -8 + 16.004 - 8.004 + 1.101 = 1.101$, and $f(-3) = -27 + 36.009 - 12.006 + 1.101 = -1.896$. Since $f(-2)$ and $f(-3)$ have opposite signs, there is a solution in the interval $(-3, -2)$.

2. 1.1 #4ab. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

   (a) $f(x) = (2 - e^x + 2x)/3, \qquad [0, 1]$.

   Solution: We compute $f'(x) = (2 - e^x)/3$, so the critical point is $x = \ln 2$. So we just need to check $|f(0)| = \frac{1}{3} \approx 0.33$, $|f(\ln 2) = \frac{2}{3} \ln 2 \approx 0.46$, and $|f(1)| = \frac{4-e}{3} \approx 0.43$. So

   $$\max_{0 \le x \le 1} f(x) = \frac{2 \ln 2}{3} \approx 0.46.$$

   (b) $f(x) = (4x - 3)/(x^2 - 2x), \qquad [0.5, 1]$.

   Solution: As before, we compute

   $$f'(x) = -\frac{2(2x^2 - 3x + 3)}{x^2(x - 2)^2}.$$

   The quadratic part is $2x^2 - 3x + 3 = 2(x - 3/4)^2 + 15/8$, so it's never zero. Hence f' is always negative, so there are no critical points. We just need to check the endpoints: $|f(0.5)| = \frac{4}{3} \approx 1.33$ and $|f(1)| = |-1| = 1$. So the maximum is

   $$\max_{0 \le x \le 1} f(x) = \frac{4}{3} \approx 1.33.$$

3. 1.1 #11. Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1) \ln x$ about $x_o = 1$.

Solution: We compute

$$\begin{aligned} f'(x) &= \ln x + 1 - \frac{1}{x} \\ f''(x) &= \frac{1}{x} + \frac{1}{x^2} \\ f'''(x) &= -\frac{1}{x^2} - \frac{2}{x^3} \\ f^{iv}(x) &= \frac{2}{x^3} + \frac{6}{x^4}. \end{aligned}$$

In particular

$$P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3.$$

(a) $P_3(0.5) = 0.5^2 - 0.5^4 = 0.1875$, while $f(0.5) \approx 0.3466$. So the actual error is $\varepsilon_{\text{actual}} = 0.16$. The theoretical error is

$$|R_3(0.5)| = \frac{f^{iv}(\xi)}{4!}(0.5-1)^4 = 0.0026(2\xi^{-3} + 6\xi^{-4}),$$

where ξ is some number between 0.5 and 1. Since $f^{iv}(\xi)$ is obviously a decreasing function, the worst case scenario is $\xi = 0.5$, which gives

$$|R_3(0.5)| \leq 0.29.$$

(b) For this question, we're asked the same thing as in part (a), except the explicit value 0.5 is replaced with the unknown value $x \in [0.5, 1.5]$. This only affects two things: we use $|x-1|^4 \leq \sup_{0.5 \leq x \leq 1.5} |x-1|^4 = 0.5^4$, and we use $|f^{iv}(\xi)| \leq \sup_{0.5 \leq \xi \leq 1.5} |f^{iv}(\xi)| = f^{iv}(0.5)$. Coincidentally, we get the same error prediction as before:

$$|R_3(x)| \leq 0.29 \text{ for any } x \in [0.5, 1.5].$$

(c) The actual value of the integral is

$$\int_{0.5}^{1.5} (x-1)\ln x \, dx = 0.088$$

using an integration by parts. The approximate value is

$$\int_{0.5}^{1.5} (x-1)^2 - \frac{1}{2}(x-1)^3 \, dx = 0.083.$$

(d) The actual error is 0.005. The upper bound from integrating the remainder is

$$\int_{0.5}^{1.5} |R_3(x)| \, dx \leq \frac{f^{iv}(0.5)}{4!} \int_{0.5}^{1.5} (x-1)^4 \, dx = 0.058.$$

(Notice that we have to use the worst possible value of $f^{iv}(\xi)$, pulling it out of the integral, rather than using something simpler like $\xi = x$.)

4. 1.1 #19. Let $f(x) = e^x$ and $x_0 = 0$. Find the n^{th} Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

   Solution: Clearly all derivatives of $f(x)$ will be $f^{(n)}(x) = e^x$, so the n^{th} Taylor polynomial is

   $$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!},$$

   while the error term is

   $$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} = \frac{e^\xi}{(n+1)!}x^{n+1}.$$

   We want this to be valid for all $x \in [0, 0.5]$, so we use $x^{n+1} \leq 0.5^{n+1}$, and $e^\xi \leq e^{0.5}$ (since the exponential function is increasing).

   Hence we have

   $$R_n(x) \leq \frac{e^{0.5}(0.5)^{n+1}}{(n+1)!},$$

   and we want this to be smaller than 10^{-6}. The only good way to do this is trial and error. Eventually we find that when $n = 7$, we have

   $$R_7(x) \leq \frac{e^{0.5}(0.5)^8}{8!} < 2 \times 10^{-7}.$$