

Math 4650 Homework #12 Solutions

6.3, #7b. Show that the product of two  $n \times n$  upper triangular matrices is upper triangular. (A full proof would be nice, but if you can't figure it out, just check the  $3 \times 3$  case.)

**Solution:** From the definition,  $A = (a_{ij})$  and  $B = (b_{ij})$  are upper triangular if and only if  $a_{ij} = 0$  for  $i > j$  and  $b_{ij} = 0$  for  $i > j$ . Let  $C = AB$ .

Then

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

We want to prove that if  $i > j$ , then  $c_{ij} = 0$ . We know that when  $i > k$  that  $a_{ik} = 0$ , and that when  $k > j$  that  $b_{kj} = 0$ . Now for every  $k$  from 1 to  $n$ , either  $k \leq j$  or  $j < k < i$  or  $k \geq i$ . (Since  $i > j$ , this is every possibility.) If  $k \leq j$  then also  $k < i$  so  $a_{ik} = 0$ . If  $j < k < i$  then  $a_{ik} = 0$  and also  $b_{kj} = 0$ . If  $k \geq i$  then also  $k > j$  so  $b_{kj} = 0$ . Thus for every  $k$ , we have  $a_{ik}b_{kj} = 0$ , so that

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = 0 \text{ if } i > j.$$

In the  $3 \times 3$  case it would be

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ 0 & a_{22}b_{22} & a_{22}b_{23} + a_{23}b_{33} \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}.$$

6.4, #6. Find all values of  $\alpha$  that make the following matrix singular.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & \alpha & 1 \\ 2 & \alpha & -1 \end{bmatrix}.$$

Just compute the  $3 \times 3$  determinant:

$$\det A = -\alpha + 4 - \alpha - \alpha + 2\alpha + 2 = 6 - \alpha = 0.$$

So the matrix is singular if and only if  $\alpha = 6$ .

6.5, #1a. Solve the following linear system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

**Solution:** This is in the form  $LUx = b$ , so we solve it by solving  $Ly = b$  and  $Ux = y$ . In other words, we first solve the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Back-substitution yields  $y_1 = 2$ ,  $y_2 = -2y_1 - 1 = -5$ , and  $y_3 = y_1 + 1 = 3$ .

Then we solve the system

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Back-substitution yields  $x_3 = 3/3 = 1$ ,  $x_2 = (-x_3 - 5)/(-2) = 3$ , and  $x_1 = (x_3 - 3x_2 + 2)/2 = -3$ .

6.5, #6a. Factor the following matrices into the  $LU$  decomposition using the  $LU$  Factorization Algorithm with  $l_{ii} = 1$  for all  $i$ .

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

**Solution:** Observe that standard Gaussian elimination will always lead to an  $LU$  decomposition, as long as you remember the multipliers and put them in the  $L$  matrix, and you never have to switch rows. (Which is the case here.)

Here I follow the algorithm in the book exactly. First observe that if  $l_{11} = 1$ , then  $u_{11} = 1$ . (Step 1.)

In Step 2, we go from  $j = 2$  to  $j = 3$ . When  $j = 2$ , we have  $u_{12} = a_{12}/l_{11} = -1$  and  $l_{21} = a_{21}/u_{11} = 2$ . When  $j = 3$ , we have  $u_{13} = a_{13}/l_{11} = 0$  and  $l_{31} = a_{31}/u_{11} = -1$ .

Now we do an  $i$  loop, from  $i = 2$  to  $i = 2$  (just one time). (Step 3.) Since  $l_{22} = 1$ , we want (Step 4)

$$u_{22} = a_{22} - \sum_{k=1}^{2-1} l_{2k}u_{k2} = a_{22} - l_{21}u_{12} = 2 + 2 = 4.$$

Now in Step 5, we do a  $j$  loop, from  $j = i + 1 = 3$  to  $j = n = 3$  (again just one step). We set

$$u_{23} = \frac{1}{l_{22}} \left[ a_{23} - \sum_{k=1}^{2-1} l_{2k}u_{k3} \right] = [a_{23} - l_{21}u_{13}] = 3 - 2 \cdot 0 = 3,$$

and

$$l_{32} = \frac{1}{u_{22}} \left[ a_{32} - \sum_{k=1}^{2-1} l_{3k}u_{k2} \right] = \frac{1}{u_{22}} [a_{32} - l_{31}u_{12}] = \frac{1}{4} [3 - (-1)(-1)] = \frac{1}{2}.$$

Finally in Step 6, we set  $l_{33} = 1$  and

$$u_{33} = a_{33} - \sum_{k=1}^{3-1} l_{nk}u_{kn} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 2 - 0 - 3 \cdot \frac{1}{2} = \frac{1}{2}.$$

The final decomposition is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Since we've almost certainly made a mistake, we do an easy check:

$$LU = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{bmatrix} = A.$$

(When computing this originally, I made three mistakes; checking the product  $LU = A$  is an easy way to realize this to fix them.)

6.5, #9a. Obtain factorizations of the form  $A = P^tLU$  for the following matrix.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Clearly we cannot even begin Gaussian elimination without switching the first two rows. Once we switch the rows using

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we get

$$PA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} = U,$$

where the only operation we had to do was add half of row two to row three (corresponding to  $l_{23} = -\frac{1}{2}$ ).

Then the decomposition is  $PA = LU$ , so that

$$A = P^tLU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}.$$

6.6, #1. Determine which of the following matrices are (i) symmetric, (ii) singular, (iii) strictly diagonally dominant, (iv) positive definite.

**a.**  $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

**b.**  $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{bmatrix}$

**c.**  $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$

**d.**  $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$

**Solution:**

- (i) Only (a) is symmetric.
- (ii) The determinant of (a) is 5, that of (b) is 24, that of (c) is  $-28 - 18 + 28 + 18 = 0$ , and that of (d) is 28. So only (c) is singular.
- (iii) The strictly diagonally dominant ones are (a) and (b).
- (iv) Only (a) has a possibility of being positive definite, but to check we should actually compute the eigenvalues. They come from

$$\lambda^2 - 6\lambda + 5 = 0,$$

whose solutions are  $\lambda = 1$  and  $\lambda = 5$ . Since all the eigenvalues are strictly positive, the matrix is positive-definite.