Math 4650 Homework #12 Solutions

1. (6.5, #1a): Solve the following linear system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution: This is in the form $LUx = b$, so we solve it by solving $Ly = b$ and $Ux = y$. In other words, we first solve the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Back-substitution yields $y_1 = 2$, $y_2 = -2y_1 - 1 = -5$, and $y_3 = y_1 + 1 = 3$.

Then we solve the system

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Back-substitution yields $x_3 = 3/3 = 1$, $x_2 = (-x_3 - 5)/(-2) = 3$, and $x_1 = (x_3 - 3x_2 + 2)/2 = -3$.

2. (6.5, #5a): Factor the following matrices into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$$

Solution: Observe that standard Gaussian elimination will always lead to an LU decomposition, as long as you remember the multipliers and put them in the L matrix, and you never have to switch rows. (Which is the case here.)

Here I follow the algorithm in the book exactly. First observe that if $l_{11} = 1$, then $u_{11} = 2$. (Step 1.)

In Step 2, we go from $j = 2$ to $j = 3$. When $j = 2$, we have $u_{12} = a_{12}/l_{11} = -1$ and $l_{21} = a_{21}/u_{11} = \frac{3}{2}$. When $j = 3$, we have $u_{13} = a_{13}/l_{11} = 1$ and $l_{31} = a_{31}/u_{11} = \frac{3}{2}$.

Now we do an i loop, from $i = 2$ to $i = 2$ (just one time). (Step 3.) Since $l_{22} = 1$, we want (Step 4)

$$u_{22} = a_{22} - \sum_{k=1}^{2-1} l_{2k}u_{k2} = a_{22} - l_{21}u_{12} = 3 - \tfrac{3}{2}(-1) = \tfrac{9}{2}.$$

Now in Step 5, we do a j loop, from $j = i + 1 = 3$ to $j = n = 3$ (again just one step). We set

$$u_{23} = \frac{1}{l_{22}} \left[a_{23} - \sum_{k=1}^{2-1} l_{2k} u_{k3} \right] = [a_{23} - l_{21} u_{13}] = [9 - \tfrac{3}{2} \cdot 1] = \tfrac{15}{2},$$

and

$$l_{32} = \frac{1}{u_{22}} \left[a_{32} - \sum_{k=1}^{2-1} l_{3k} u_{k2} \right] = \tfrac{1}{u_{22}} [a_{32} - l_{31} u_{12}] = \tfrac{2}{9} [3 - \tfrac{3}{2}(-1)] = 1.$$

Finally in Step 6, we set $l_{33} = 1$ and

$$u_{33} = a_{33} - \sum_{k=1}^{3-1} l_{nk} u_{kn} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = 5 - \tfrac{3}{2} \cdot 1 - 1 \cdot \tfrac{15}{2} = -4.$$

The final decomposition is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & -4 \end{bmatrix}.$$

Since we've almost certainly made a mistake, we do an easy check:

$$LU = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} = A.$$

(When computing this originally, I made three mistakes; checking the product $LU = A$ is an easy way to realize this to fix them.)

3. (6.6, #3a): Use the LDL^t Factorization Algorithm to find a factorization of the form $A = LDL^t$ for the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: We could do this by just using the standard LU factorization (in other words, doing Gaussian elimination and remembering the multipliers in L). Since the matrix is symmetric, U will always be in the form $U = DL^t$ for some diagonal matrix D. Here I'll do it both ways (using Gaussian elimination and using the book's algorithm).

- (Gaussian). Immediately we see that $m_{21} = -\frac{1}{2}$ and $m_{31} = 0$, since those are the factors we need to clear out the first column. The operations $R_2 \to R_2 - m_{21} R_1$ and $R_3 \to R_3 - m_{31} R_1$ lead to

$$A^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

The last thing to do is clear out the second column, and the required factor is $m_{32} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$. We finally get

$$U = A^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix},$$

with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}.$$

Again a simple matrix multiplication verifies that $LU = A$.

The diagonal matrix D is formed from the diagonal elements of U, so that

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

Although we already checked $LU = A$, if we wanted to, we could have instead checked that $LDL^t = A$:

$$LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ 0 & -1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = A$$

- (Algorithm). Step 1: for i from 1 to 3.

  When $i = 1$, we go into Step 2, which says j should go from 1 to 0. This is impossible, so we don't execute anything. In Step 3, we set $d_1 = a_{11} - \sum_{j=1}^{0} l_{1j} v_j = a_{11} = 2$. In Step 4, we go from $j = 2$ to $j = 3$. Observe that we again have a sum $\sum_{k=1}^{0}$, which is just zero. So we get

  $$l_{21} = a_{21}/d_1 = -\frac{1}{2} \qquad \text{and} \qquad l_{31} = a_{31}/d_1 = 0.$$

  Now back to Step 1, when $i = 2$. Now in Step 2, we go from $j = 1$ to $j = 1$ and get $v_1 = l_{21}d_1 = -1$. In Step 3, we set $d_2 = a_{22} - l_{21}v_1 = 2 - (-\frac{1}{2})(-1) = \frac{3}{2}$. In Step 4, we go from $j = 3$ to $j = 3$ and get

  $$l_{32} = (a_{32} - l_{31}v_1)/d_2 = (-1 - (0)(-1))/(3/2) = -\frac{2}{3}.$$

  Now one more time to Step 1, when $i = 3$. In Step 2, we go from $j = 1$ to $j = 2$, and get $v_1 = l_{31}d_1 = 0$ and $v_2 = l_{32}d_2 = -1$. In Step 3, we set $d_3 = a_{33} - l_{31}v_1 - l_{32}v_2 = 2 - 0 - (-\frac{2}{3})(-1) = \frac{4}{3}$. Finally Step 4 is never executed, since j is going from 4 to 3.

  We finally have

  $$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

4. (6.6, #5a): Use the Cholesky Algorithm to find a factorization of the form $A = LL^t$ for the following matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution: Again there are two methods. One is to use Gaussian elimination to get the LDL^t algorithm as in problem #3. The other is to use the book's algorithm, which works the same way as above (so I'll skip it).

Following the same procedure as above, we obtain $A = LDL^t$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

Then the Cholesky factorization is $(L')(L')^t$, where $L' = L\sqrt{D}$.

Thus we have

$$L' = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} \end{bmatrix}$$

Finally we can check that $(L')(L')^t = A$:

$$(L')(L')^t = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \sqrt{\frac{4}{3}} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A$$