

Math 4650 Homework #11 Solutions

1. (6.1, #3): Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = 1$, $x_2 = -1$, $x_3 = 3$.)

(a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 4x_3 &= 11 \end{aligned}$$

Solution: Following the algorithm exactly, we first have to multiply equation (1) by 0.5 and subtract that from equation (2); then multiply equation (1) by 0.25 and subtract that from equation (3):

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 5.5x_2 + 1.5x_3 &= -1 \\ 2.3x_2 + 3.8x_3 &= 9 \end{aligned}$$

(We have to round 2.25 to 2.3 and 3.75 to 3.8.)

Next we need to multiply equation (2) by $2.3/5.5 = 0.42$ and subtract that from equation (3). We get:

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 5.5x_2 + 1.5x_3 &= -1 \\ 3.2x_3 &= 9.4 \end{aligned}$$

Now we can solve using back-substitution:

$$\begin{aligned} x_3 &= 9.4/3.2 = 2.9 \\ x_2 &= (-1 - 1.5 \cdot 2.9)/(5.5) = -0.98 \\ x_1 &= (8 - 2.9 - 0.98)/4 = 4.1/4 = 1.0 \end{aligned}$$

These results are not far from the true values.

(b)

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solution: We follow the same procedure. The first two steps are identical:

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 3.5x_2 - 2x_3 &= -9.5 \\ 0.75x_2 - 3.5x_3 &= -11 \end{aligned}$$

Now we have to compute $0.75/3.5 = 0.21$; we subtract 0.21 times equation (2) from equation (3).

$$\begin{aligned}4x_1 + x_2 + 2x_3 &= 9 \\3.5x_2 - 2x_3 &= -9.5 \\-3.1x_3 &= -9\end{aligned}$$

Finally we solve using back-substitution:

$$\begin{aligned}x_3 &= 2.9 \\x_2 &= (-9.5 + 2 \cdot 2.9)/(3.5) = -1.1 \\x_1 &= (9 + 1.1 - 2 \cdot 2.9)/4 = 1.1\end{aligned}$$

2. (6.1, #15)

(a) Show that the Gauss-Jordan method requires

$$\frac{n^3}{2} + n^2 - \frac{n}{2} \text{ multiplications/divisions and } \frac{n^3}{2} - \frac{n}{2} \text{ additions/subtractions.}$$

Solution:

The algorithm for Gaussian elimination is as follows.

```
for k from 1 to n-1 do
  for i from k+1 to n do
    m = a[i,k] / a[k,k]
    for j from k+1 to n do
      a[i,j] = a[i,j] - m*a[k,j]
    end do
    b[i] = b[i] - m*b[k]
  end do
end do
```

The difference in Gauss-Jordan is that instead of just clearing out the rows below the k^{th} , we clear out *all* rows except the k^{th} at each step. Other than that, it's the same algorithm. So the only difference will be that instead of i going from k to n , i will go from 1 to n , skipping only k . Also, instead of k going from 1 to $n-1$, k must go from 1 to n ; this is because in Gaussian elimination, we didn't do anything with the last column, but here we want to clear that out as well. (The j -loop does not change, since everything to the left of the k column will already be cleared out.) Here then is the Gauss-Jordan algorithm.

```
for k from 1 to n do
  for i from 1 to n do
    if (i <> k) then
      m = a[i,k] / a[k,k]
      for j from k+1 to n do
```

```

        a[i,j] = a[i,j] - m*a[k,j]
    end do
    b[i] = b[i] - m*b[k]
end if
end do
end do

```

Now we count how many operations this requires. Inside the j -loop, we do one multiplication/division at each j -step, and therefore $(n - k)$ in total through the j -loop. With the two extra multiplication/divisions before and after the j -loop, there is a total of $(n - k + 2)$ computations each time through the i -loop. We go through the i -loop $(n - 1)$ times (no computations are done if $i = k$), so that we have $(n - 1)(n - k + 2)$ multiplications each time through the k -loop. Finally we have the n divisions to actually solve for the x_i .

The total number of multiplications/divisions is then

$$n + \sum_{k=1}^n (n - 1)(n - k + 2) = n + \frac{n^3}{2} + n^2 - \frac{3n}{2} = \frac{n^3}{2} + n^2 - \frac{n}{2}.$$

Similarly, the number of additions/subtractions is

$$\sum_{k=1}^n (n - 1)(n - k + 1) = \frac{n^3}{2} - \frac{n}{2}.$$

(b) The table appears below.

n	G-E m/d	G-E a/s	G-J m/d	G-J a/s
3	17	11	21	12
10	430	375	595	495
50	44,150	42,875	64,975	62,475
100	343,300	338,250	509,950	499,950

Gauss-Jordan consistently requires more computation.

3. (6.2 #11a) Use Gaussian elimination and three-digit rounding arithmetic to solve the following linear systems, and compare the approximations to the actual solution.

$$0.03x_1 + 58.9x_2 = 59.2$$

$$5.31x_1 - 6.10x_2 = 47.0$$

Actual solution [10,1].

Solution:

We compute $5.31/0.03 = 177$, so that the equations reduce to

$$0.03x_1 + 58.9x_2 = 59.2$$

$$-10,400x_2 = -10,500$$

Thus we get $x_2 = 1.01$, and

$$x_1 = (59.2 - 58.9 \cdot 1.01)/0.03 = (59.2 - 59.5)/(0.03) = -10.0.$$

The value for x_1 is quite far from the correct answer, although x_2 is pretty close.

4. (6.2 #13a) Repeat the previous problem using Gaussian elimination with partial pivoting.

Solution: Comparing 0.03 to 5.31, obviously the second is much larger, so we should switch rows. We get

$$5.31x_1 - 6.10x_2 = 47.0$$

$$0.03x_1 + 58.9x_2 = 59.2$$

Now we compute $0.03/5.31 = 0.00565$, so we obtain

$$5.31x_1 - 6.10x_2 = 47.0$$

$$58.9x_2 = 58.9$$

Hence we end up with $x_2 = 1.00$ and

$$x_1 = (47.0 + 6.10 \cdot 1.00)/(5.31) = 10.0.$$

Partial pivoting does much better.

5. (6.2 #17a) Repeat the previous problem using Gaussian elimination with scaled partial pivoting.

Solution: Instead of comparing 0.03 with 5.31, we first find the maximum of the first row: 59.2. We then find the maximum of the second row: 47.0. We now compare the scaled versions of the coefficients:

$$0.03/59.2 = 0.000507 \quad \text{vs.} \quad 5.31/47.0 = 0.113.$$

Again we find the second row to have a larger pivot element, so we should switch rows.

Since we get the exact same equations as in #13a, the result is exactly the same: $x_2 = 1.00$ and $x_1 = 10.0$.

6. (6.3 #3) Given the two 4×4 linear systems having the same coefficient matrix:

$$\begin{array}{ll} x_1 - x_2 + 2x_3 - x_4 = 6, & x_1 - x_2 + 2x_3 - x_4 = 1 \\ x_1 - x_3 + x_4 = 4, & x_1 - x_3 + x_4 = 1 \\ 2x_1 + x_2 + 3x_3 - 4x_4 = -2, & 2x_1 + x_2 + 3x_3 - 4x_4 = 2 \\ -x_2 + x_3 - x_4 = 5, & -x_2 + x_3 - x_4 = -1 \end{array}$$

(a) Solve the linear system by applying Gaussian elimination to the augmented matrix

$$\left[\begin{array}{cccc|cc} 1 & -1 & 2 & -1 & 6 & 1 \\ 1 & 0 & -1 & 1 & 4 & 1 \\ 2 & 1 & 3 & -4 & -2 & 2 \\ 0 & -1 & 1 & -1 & 5 & -1 \end{array} \right]$$

Solution:

$$\begin{aligned} & \left[\begin{array}{cccc|cc} 1 & -1 & 2 & -1 & 6 & 1 \\ 1 & 0 & -1 & 1 & 4 & 1 \\ 2 & 1 & 3 & -4 & -2 & 2 \\ 0 & -1 & 1 & -1 & 5 & -1 \end{array} \right] \sim \left[\begin{array}{cccc|cc} 1 & -1 & 2 & -1 & 6 & 1 \\ 0 & 1 & -3 & 2 & -2 & 0 \\ 0 & 3 & -1 & -2 & -14 & 0 \\ 0 & -1 & 1 & -1 & 5 & -1 \end{array} \right] \\ & \sim \left[\begin{array}{cccc|cc} 1 & -1 & 2 & -1 & 6 & 1 \\ 0 & 1 & -3 & 2 & -2 & 0 \\ 0 & 0 & 8 & -8 & -8 & 0 \\ 0 & 0 & -2 & 1 & 3 & -1 \end{array} \right] \sim \left[\begin{array}{cccc|cc} 1 & -1 & 2 & -1 & 6 & 1 \\ 0 & 1 & -3 & 2 & -2 & 0 \\ 0 & 0 & 8 & -8 & -8 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array} \right] \end{aligned}$$

And thus the solutions to the first system are

$$x_4 = -1, \quad x_3 = -2, \quad x_2 = -6, \quad x_1 = 3,$$

while the solutions to the second system are

$$x_4 = 1, \quad x_3 = 1, \quad x_2 = 1, \quad x_1 = 1.$$

(b) Solve the linear systems by finding and multiplying by the inverse of

$$A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \\ 2 & 1 & 3 & -4 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

Solution: We solve this using row reduction, where the right side is the identity.

$$\begin{aligned} & \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & -4 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 3 & -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 8 & -8 & 1 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 \end{array} \right] \\ & \sim \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 \end{array} \right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -1 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 \end{array} \right] \end{aligned}$$

Now with the inverse matrix, we multiply by the given vector:

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ -8 & -3 & -8 & -1 \\ 3 & 5 & -1 & -1 \\ 4 & 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ -8 & -3 & -8 & -1 \\ 3 & 5 & -1 & -1 \\ 4 & 4 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Both methods are a giant pain in the ass, but Gaussian elimination is a lot quicker.

7. (6.4 #1ab) Use Definition 6.14 to compute the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution: Expand on the first row (since it has a zero). We get

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = (1) \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + (-1)(2) \cdot \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 0$$

$$= 1(1 \cdot 1 - (-1) \cdot 1) - 2(2 \cdot 1 - (-1) \cdot 3)$$

$$= -8$$

(b) $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$

Solution: Again expand on the first row. We get

$$\begin{vmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \end{vmatrix} = 4 \cdot \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 4(1 \cdot 3 - 0 \cdot 2) + 1(2 \cdot 2 - 1 \cdot 2)$$

$$= 14$$

8. (6.4 #3ab) Repeat the previous problem using row reduction first.

(a) $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & -5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & \frac{8}{3} \end{vmatrix} = -8.$

(b) $\begin{vmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & \frac{5}{2} \end{vmatrix} = \begin{vmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{vmatrix} = 14.$