

Math 4650 Homework #11 Solutions

1. Verify that the difference equation has characteristic roots given by $1 \pm \sqrt{3}i = 2e^{\pm i\pi/3}$. Then check that $w_k = 2^k \cos(k\pi/3)$ and $w_k = 2^k \sin(k\pi/3)$ are both solutions.

$$w_{k+1} = 2w_k - 4w_{k-1}.$$

The characteristic equation is

$$\lambda^2 = 2\lambda - 4,$$

which factors as $(\lambda - 1)^2 + 3 = 0$, so that the solutions are

$$\lambda = 1 \pm \sqrt{3}i.$$

Putting this in polar form $re^{i\theta}$, we have $r^2 = a^2 + b^2 = 4$, so that $r = 2$, and $\theta = \arctan \sqrt{3} = \frac{\pi}{3}$.

Now we check the given solutions. Try $w_k = 2^k \cos(\frac{k\pi}{3})$: then

$$w_{k+1} = 2^{k+1} \cos(\frac{k\pi}{3} + \frac{\pi}{3}) = 2^{k+1} \cos \frac{k\pi}{3} \cos \frac{\pi}{3} - 2^{k+1} \sin \frac{k\pi}{3} \sin \frac{\pi}{3} = 2^k \cos \frac{k\pi}{3} - \sqrt{3}2^k \sin \frac{k\pi}{3},$$

while

$$\begin{aligned} 2w_k - 4w_{k-1} &= 2^{k+1} \cos \frac{k\pi}{3} - 2^{k+1} \cos(\frac{k\pi}{3} - \frac{\pi}{3}) \\ &= 2^{k+1} \cos \frac{k\pi}{3} - 2^{k+1} \cos \frac{k\pi}{3} \cos \frac{\pi}{3} - 2^{k+1} \sin \frac{k\pi}{3} \sin \frac{\pi}{3} \\ &= 2^{k+1} \cos \frac{k\pi}{3} - 2^k \cos \frac{k\pi}{3} - \sqrt{3}2^k \sin \frac{k\pi}{3}. \end{aligned}$$

Thus the equation is satisfied. The other one works the same way.

2. Verify that the difference equation has repeated characteristic roots given by 3, 3. Then check that $w_k = 3^k$ and $w_k = k3^k$ are both solutions.

$$w_{k+1} = 6w_k - 9w_{k-1}.$$

The characteristic equation is $0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. So the root $\lambda = 3$ is repeated.

Plugging in the given solution $w_k = 3^k$ we get

$$6w_k - 9w_{k-1} = 6 \cdot 3^k - 9 \cdot 3^{k-1} = 2 \cdot 3^{k+1} - 3^{k+1} = 3^{k+1} = w_{k+1}.$$

Next trying $w_k = k3^k$ we get

$$6w_k - 9w_{k-1} = 6k3^k - 9(k-1) \cdot 3^{k-1} = 2k3^{k+1} - (k-1)3^{k+1} = k3^{k+1} + 3^{k+1} = (k+1)3^{k+1} = w_{k+1}.$$

So both suggested terms are solutions.

3. Find the solution of the difference equation

$$w_{k+1} = w_k + 6w_{k-1}, \quad w_0 = 4, \quad w_1 = -3$$

The characteristic equation is

$$\lambda^2 - \lambda - 6 = 0 = (\lambda - 3)(\lambda + 2).$$

So the general solution is

$$w_k = A3^k + B(-2)^k.$$

Now plugging in $k = 0$ we get $A + B = 4$, and plugging in $k = 1$ we get $3A - 2B = -3$. Solving these, we get $A = 1$, $B = 3$.

So the solution is

$$w_k = 3^k + 3(-2)^k.$$

4. For the difference method

$$w_{k+1} = 3w_{k-1} - 2w_{k-2} + 3hf(t_k, w_k) - 3hf(t_{k-1}, w_{k-1}),$$

(a) Find the local truncation error. (Hint: use the trick from class: replace $f(t - h, y(t - h))$ with $y'(t - h)$.)

We assume $w_k = y(t_k)$, $w_{k-1} = y(t_{k-1}) = y(t_k - h)$, and $w_{k-2} = y(t_{k-2}) = y(t_k - 2h)$. Then we have

$$\begin{aligned} w_{k+1} &= 3y(t - h) - 2y(t - 2h) + 3hy'(t) + 3hy'(t - h) \\ &= 3 \left(y(t) - hy'(t) + \frac{h^2}{2}y''(t) - \frac{h^3}{6}y'''(t) \right) \\ &\quad - 2 \left(y(t) - 2hy'(t) + 2h^2y''(t) - \frac{4h^3}{3}y'''(t) \right) + 3hy'(t) \\ &\quad - 3h \left(y'(t) - hy''(t) + \frac{h^2}{2}y'''(t) \right) + O(h^4) \\ &= y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{2h^3}{3}y'''(t) + O(h^4). \end{aligned}$$

On the other hand we have

$$y(t_{k+1}) = y(t_k + h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + \frac{h^3}{6}y'''(t) + O(h^4).$$

Thus the local truncation error is

$$\tau_k = \frac{y(t_{k+1}) - w_{k+1}}{h} = -\frac{h^2}{2}y'''(t).$$

(b) Is the method zero-stable? Justify your answer.

The characteristic polynomial is

$$\lambda^3 - 3\lambda + 2 = 0,$$

which factors as

$$(\lambda - 1)^2(\lambda + 2) = 0,$$

so the roots are $\lambda = 1$, $\lambda = 1$, and $\lambda = -2$. The method is therefore unstable, since it has both a repeated root of size one, and a root of size greater than one in absolute value.

5. Graph the region of absolute stability for (the forward) Euler's method.

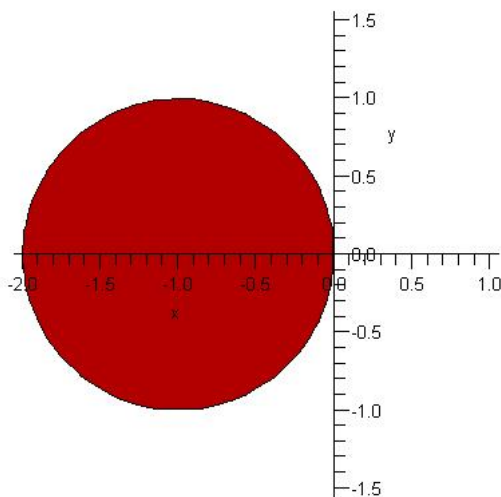
The test equation is $y' = \lambda y$, and Euler's method is

$$w_{k+1} = w_k + hf(t_k, w_k) = w_k + h\lambda w_k = (1 + h\lambda)w_k.$$

Thus the stability function is $Q(z) = 1 + z$, and the region of absolute stability is $|Q(z)| < 1$. Now

$$|Q(z)|^2 = |1 + x + iy|^2 = (1 + x)^2 + y^2,$$

so that $R = \{z : |Q(z)|^2 < 1\}$ is the interior of the circle centered at $(-1, 0)$ with radius one, i.e., the region in red below.



6. Graph the region of absolute stability for the implicit (backward) Euler's method. Use this to show the implicit Euler method is A-stable.

As above, the test equation is $y' = \lambda y$, and the implicit Euler method is

$$w_{k+1} = w_k + hf(t_{k+1}, w_{k+1}) = w_k + h\lambda w_{k+1}.$$

Solving for w_{k+1} we get

$$w_{k+1} = \frac{1}{1 - h\lambda} w_k.$$

So the stability function is $Q(z) = \frac{1}{1-z}$. The region of absolute stability consists of $\frac{1}{|1-z|} < 1$, or solving, $|1 - z| > 1$.

Now

$$|1 - z|^2 = |1 - x - iy|^2 = (1 - x)^2 + y^2 > 1$$

is the *exterior* of the circle with center $(1, 0)$ and radius one, the region in red below. Since this region includes the entire left half-plane, the implicit Euler method is A-stable.

