Math 4650 Homework #10 Solutions

1. For the differential equation $y' = 1 + (t - y)^2$, $2 \le t \le 3$, $y(2) = 1$ with $h = 0.5$, use the two-step Adams-Bashforth method and the one-step Adams-Moulton method (the trapezoid rule) to estimate $y(3)$ by hand. Use the midpoint method to get your starting values. Compare to the actual answer $y(3) = 2.5$, and also compare to the answers you got using the methods from the last homework. Which is the best second-order method for this problem?

   Solution: For both methods we only need two starting values: $w_0 = 1$ and w_1, which comes from the midpoint method

$$\begin{aligned} w_1 &= w_0 + hf(t + h/2, w_0 + h/2f(t, w_0)) \\ &= 1 + 0.5f(2.25, 1 + 0.25f(2, 1)) \\ &= 1 + 0.5f(2.25, 1.5) \\ &= 1 + 0.5(1 + (0.75)^2) \\ &= 1.78125. \end{aligned}$$

   For the Adams-Bashforth method, we have

$$\begin{aligned} w_2 &= w_1 + \frac{h}{2}\left(3f(t_1, w_1) - f(t_0, w_0)\right) \\ w_2 &= 1.78125 + 0.25\left(3f(2.5, 1.78125)) - f(2, 1)\right) \\ &= 2.418701172. \end{aligned}$$

   For the one-step Adams-Moulton method, we have to solve the equation

$$w_2 = w_1 + \frac{h}{2}\left(f(t_1, w_1) + f(t_2, w_2)\right).$$

   It's easiest to just expand everything out using the numbers we already have (since we only need one step of it).

$$w_2 = 1.78125 + 0.25(1.516601562 + 1 + (3 - w_2)^2)$$

   which reduces to

$$0.25w_2^2 - 2.50w_2 + 4.660400390 = 0,$$

   with solutions

$$w_2 = 2.478413507 \qquad \text{and} \qquad w_2 = 7.521586493.$$

   The one we want is clearly the first one.

   Not too surprisingly, considering what was said in class about this, Adams-Moulton is better. (It's *always* better than the corresponding Adams-Bashforth method, if you can solve the equation.)

Several people asked how you'd know which solution to pick in general. Essentially what you'd have to do is solve the equation generally; writing $C = f(t_i, w_i)$, the equation is

$$w_{i+1} = w_i + \frac{h}{2}\big(C + 1 + (t_{i+1} - w_{i+1})^2\big),$$

and solving this you'd get

$$w_{i+1} = \frac{1 + ht \pm \sqrt{1 + 2ht - 2hw_i - h^2C - h^2}}{h}.$$

The one you want is the one that, when you plug in $h = 0$, gives you something reasonable out. Clearly if you take the plus sign, as $h \to 0$ the limit is $\frac{2}{0} = \infty$, so the plus sign can't be right. The minus sign at least gives you $\frac{0}{0}$, so there's some hope there. If you actually compute this limit, you get w_i, which is what you'd want and expect. That means the minus sign is *always* the right choice for this particular problem.

In general this issue doesn't come up because you'd hardly ever use the Adams-Moulton method except in a predictor-corrector algorithm, so you wouldn't be solving the equation by hand.

2. For the differential equation $y'(t) = 2y + t^2 + t + 1$ with initial condition $y(0) = 1$, which has exact solution $y(t) = 2e^{2t} - t^2/2 - t - 1$, write programs to use the Adams-Moulton three-step method, the Adams-Bashforth four-step method, and the fourth-order Runge-Kutta method, with step size $h = 0.1$, to approximate $y(2)$. Obtain initial values for the multistep methods using Runge-Kutta of order four. Compare your answers to the true value. Which of the three does best?

   Solution: The algorithm for the fourth-order Adams-Bashforth method is:

```
h = (b-a)/N
w = alpha
t = a
% Generate w1, w2, w3 using fourth-order Runge-Kutta
k1 = f(t, w)
k2 = f(t+h/2, w+h/2*k1)
k3 = f(t+h/2, w+h/2*k2)
k4 = f(t+h, w+h*k3)
w = w + h/6*(k1+2*k2+2*k3+k4)
t = t + h
f_3 = k1
% Save values of f since we need them; this is f_{k-3} when k=3
%
k1 = f(t, w)
k2 = f(t+h/2, w+h/2*k1)
k3 = f(t+h/2, w+h/2*k2)
k4 = f(t+h, w+h*k3)
```

```
w = w + h/6*(k1+2*k2+2*k3+k4)
t = t + h
f_2 = k1
%
k1 = f(t, w)
k2 = f(t+h/2, w+h/2*k1)
k3 = f(t+h/2, w+h/2*k2)
k4 = f(t+h, w+h*k3)
w = w + h/6*(k1+2*k2+2*k3+k4)
t = t + h
f_1 = k1
f_0 = f(t, w)
for i from 4 to N do
   w = w + h/24*(-9*f_3 + 37*f_2 - 59*f_1 + 55*f_0)
   t = t + h
   f_3 = f_2
   f_2 = f_1
   f_1 = f_0
   f_0 = f(t, w)
end do
output w
```

This produces the output $w = 104.0437874$.

For the Adams-Moulton technique, we must first solve the implicit equation. The general formula is

$$w_{i+1} = w_i + \frac{9h}{24} f(t_{i+1}, w_{i+1}) + \frac{h}{24}\big[19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\big].$$

For this particular function, we have

$$w_{i+1} = w_i + \frac{18 h w_{i+1}}{24} + \frac{h}{24}\big[9(t_{i+1}^2 + t_{i+1} + 1) + 19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\big],$$

so solving for w_{i+1} we get the algorithm

$$w_{i+1} = \frac{1}{1 - 0.75h}\Big(w_i + \frac{h}{24}\big[9(t_{i+1}^2 + t_{i+1} + 1) + 19 f(t_i, w_i) - 5 f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\big]\Big).$$

The algorithm based on this method again uses fourth-order Runge-Kutta, but now only needs three previous values (so only needs to compute w_1 and w_2).

```
h = (b-a)/N
w = alpha
t = a
% Generate w1, w3 using fourth-order Runge-Kutta
```

```
k1 = f(t, w)
k2 = f(t+h/2, w+h/2*k1)
k3 = f(t+h/2, w+h/2*k2)
k4 = f(t+h, w+h*k3)
w = w + h/6*(k1+2*k2+2*k3+k4)
t = t + h
f_2 = k1 % Save values of f since we need them; this is f_{k-3} when k=3
%
k1 = f(t, w)
k2 = f(t+h/2, w+h/2*k1)
k3 = f(t+h/2, w+h/2*k2)
k4 = f(t+h, w+h*k3)
w = w + h/6*(k1+2*k2+2*k3+k4)
t = t + h
f_1 = k1
%
f_0 = f(t, w)
for i from 3 to N do
   t = t + h
   f_rump = t^2 + t + 1
   w = (w +h/24*(9*f_rump + 19 f_0 - 5 f_1 + f_2))/(1-0.75*h)
   f_2 = f_1
   f_1 = f_0
   f_0 = f_rump+2*w
end do
output w
```

This algorithm gives the result $w = 104.2100676$.

The algorithm for the fourth-order Runge-Kutta method is

```
h = (b-a)/N
w = alpha
t = a
for i from 1 to N do
   k1 = f(t,w)
   kw = f(t+h/2, w+h/2*k1)
   k3 = f(t+h/2, w+h/2*k2)
   k4 = f(t+h, w+h*k3)
   w = w + h/6*(k1+2*k2+2*k3+k4)
   t = t + h
end do
output w
```

This method gives $w = 104.1915702$.

The true value is $y(2) = 2e^4 - 5 = 104.1963001$.

The error in four-step Adams-Bashforth is $104.0437874 - 104.1963001 = -0.1525127$.

The error in three-step Adams-Moulton is $104.2100676 - 104.1963001 = 0.0137675$.

The error in fourth-order Runge-Kutta is $104.1915702 - 104.1963001 = -0.0047299$.

Although all three methods are fourth-order, the Runge-Kutta has the smallest actual error, while the third-order Adams-Moulton is second smallest, and fourth-order Adams-Bashforth is the worst.