Math 4650 Final Exam Solutions

1. True/False, and explain your answer in one sentence. Half credit for the correct circling, half credit for a correct justification.

   (a) The main advantage of an Adams-Bashforth method over a Runge-Kutta method of the same order is the fact that the error coefficient is smaller.
   T
   F
   Typically any Runge-Kutta will have noticeably better error coefficient than an Adams-Bashforth; the advantage of Adams-Bashforth is fewer function evaluations.

   (b) The determinant of a strictly diagonally dominant matrix is never zero.
   T
   F
   A strictly diagonally dominant matrix is always nonsingular, so in particular its determinant is not zero.

   (c) A difference scheme with characteristic polynomial $(\lambda - 1)^2(\lambda + 1)^2$ is weakly stable.
   T
   F
   The method does not satisfy the root condition, since it has repeated roots on the unit circle of $+1$ twice and -1 twice.

   (d) There are times when reducing step size in a consistent difference scheme can make the approximation error worse.
   T
   F
   Roundoff error is an issue with differential equations, and the practical local truncation error includes a roundoff term like $\frac{10^{-k}}{h}$.

   (e) Since $100! \approx 10^{158}$, it is impossible to compute the determinant of a 100×100 matrix before the heat death of the universe.
   T
   F
   It would be impossible to compute from the definition, but using Gaussian elimination first would reduce the number of computations to roughly $100^3 = 10^6$.

2. Consider the difference scheme

$$w_{k+1} = -6w_k + 7w_{k-1} + 7hf(t_k, w_k) + hf(t_{k-3}, w_{k-3})$$

   for approximating the solution of $y'(t) = f(t, y(t))$.

   (a) Is the method consistent?

Solution: Assume $w_k = y(t)$, $w_{k-1} = y(t-h)$, $w_{k-3} = y(t-3h)$. Then

$$\begin{aligned} w_{k+1} &= -6y(t) + 7y(t-h) + 7hy'(t-h) + hy'(t-3h) \\ &= -6y(t) + 7\big[y(t) - hy'(t) + \tfrac{h^2}{2}y''(t) - \tfrac{h^3}{6}y'''(t)\big] \\ &\qquad + 7hy'(t) + h\big[y'(t) - 3hy''(t) + \tfrac{9h^2}{2}y'''(t)\big] + O(h^4) \\ &= y(t) + hy'(t) + \tfrac{h^2}{2}y''(t) + \tfrac{10h^3}{3}y'''(t) + O(h^4). \end{aligned}$$

On the other hand

$$y(t+h) = y(t) + hy'(t) + \tfrac{h^2}{2}y''(t) + \tfrac{h^3}{6}y'''(t) + O(h^4).$$

So the local truncation error is

$$\frac{w_{k+1} - y(t_{k+1})}{h} = \frac{19h^2}{6}y''(t) + O(h^3).$$

In particular since this goes to zero as h goes to zero, the method is consistent.

(b) Is the method zero-stable?

Solution: The characteristic difference equation is $w_{k+1} + 6w_k - 7w_{k-1} = 0$, so the characteristic polynomial is

$$\lambda^2 + 6\lambda - 7 = 0.$$

The roots of this equation are $\lambda = 1$ and $\lambda = -7$. Thus the method is not zero-stable.

3. For the symmetric matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

(a) Determine the LDL^T factorization.

Solution: The first reduction has multipliers $l_{21} = 2$ and $l_{31} = 1$, so we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then the next multiplier is $l_{32} = -1$, and we get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We read off the diagonal terms to get

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check:

$$LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A.$$

(b) Is the matrix positive-definite?

Solution: Since the diagonal terms are not all positive, the matrix is not positive-definite.

4. The algorithm for factorization of a symmetric matrix from class is

```
10   for i from 1 to n do
20      D[i] = a[i,i]
30      for j from 1 to i-1 do
40         v[j] = L[i,j] * D[j]
50         D[i] = D[i] - L[i,j] * v[j]
60      end do
70      for j from i+1 to n do
80         L[j,i] = a[j,i]
90         for k from 1 to i-1 do
100           L[j,i] = L[j,i] - L[j,k] * v[k]
110        end do
120        L[j,i] = L[j,i] / D[i]
130     end do
140  end do
```

Set up and evaluate an expression for the number of multiplications/divisions in this algorithm.

Hint: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution: In the k-loop at line 100 there is one multiplication; thus from 90 to 110 we have

$$\sum_{k=1}^{i-1} 1 = i - 1.$$

One extra division at line 120 gives a total of i multiplications between 80 and 120. Summing this up from $j = i + 1$ to $j = n$ we get

$$\sum_{j=i+1}^{n} i = i \sum_{j=i+1}^{n} 1 = i(n - i)$$

as the number of multiplications between line 70 and line 130.

Now for the j-loop in lines 30–60, we have

$$\sum_{j=1}^{i-1} 2 = 2(i-1).$$

Thus between lines 20 and 120, there is a total of

$$2(i-1) + i(n-i) = -i^2 + (n+2)i - 2$$

operations.

Finally to get the entire algorithm we sum from $i = 1$ to $i = n$:

$$\sum_{i=1}^{n} -i^2 + (n+2)i - 2 = -\frac{n(n+1)(2n+1)}{6} + (n+2)\frac{n(n+1)}{2} - 2n$$
$$= \frac{1}{6}\big(- 2n^3 - 3n^2 - n + 3n^3 + 9n^2 + 6n - 12n\big) = \frac{n^3}{6} + n^2 - \frac{7n}{6}.$$

5. Consider the initial value problem

$$y' = -4y, \qquad y(0) = 1,$$

using the midpoint method.

(a) Write down and simplify the difference equation obtained from the midpoint method with $h = 1$.

Solution: The difference scheme is

$$w_{k+1} = w_k + hf\big(t_k + \tfrac{h}{2}, w_k + \tfrac{h}{2}f(t_k, w_k)\big) = w_k + \tfrac{1}{2}f\big(t_k + \tfrac{1}{2}, -w_k\big)$$
$$= w_k + 4w_k = 5w_k.$$

(b) Solve the difference equation explicitly, and solve the differential equation explicitly. How do the solutions compare?

Solution: The solution is obviously $w_k = 5^k$.

The actual solution of the differential equation is $y(t) = e^{-4t}$.

The solutions are nothing like each other (the actual solution shrinks to zero, while the difference scheme blows up to infinity).

(c) What does part (b) tell you about the A-stability of the midpoint method?

This means the midpoint scheme is not A-stable. Specifically, $h\lambda = -4$ is not in the region of absolute stability.

6. Acme brand difference scheme has local truncation error

$$E_A(h) = \frac{w_{k+1}^{(A)} - y(t_{k+1})}{h} = 10h^2 y'''(t_k) + O(h^3).$$

Brand X difference scheme has local truncation error

$$E_X(h) = \frac{w_{k+1}^{(X)} - y(t_{k+1})}{h} = -20h^2 y'''(t_k) + O(h^3).$$

(a) If for some differential equation starting at $t = 0$, when $h = 0.1$, Acme brand gives $w_{11}^{(A)} = 12.7$ and Brand X gives $w_{11}^{(X)} = 13.0$, what is the best estimate for $y(1.1)$? What is the coefficient $y'''(1)$?

Solution: The unknowns here are $y'''(1)$ and $y(1.1)$, and we have two (approximate) equations for them.

$$\begin{aligned} 13.0 - y(1) &\approx 10(0.1)^3 y'''(1) \\ 13.3 - y(1) &\approx -20(0.1)^3 y'''(1) \end{aligned}$$

$$2 \cdot 13.0 + 13.3 - 3y(1) \approx 0$$

so $y(1) \approx 13.1$.

We also get $y'''(1) = -1$.

(b) If you wanted the local truncation error for Acme brand to be $E_A(h) = 0.001$, what h should you use instead?

Solution The local Acme truncation error is $-10h^2$, and we want the absolute value of it to be 0.001, which means $h^2 = 0.0001$. So $h = 0.01$.