Math 4650 Final Exam

Name: ______________________________

Instructions:

- Do any **five** of the problems. If you try more than five, you must clearly label which five you want graded.
- Attach your formula sheet to the exam when finished.
- Show work when feasible. When using results from class, cite them clearly.
- Don't cheat.
- Stay calm, confident, and calculating correctly.

I would like the following problems to be graded (circle five):

1 2 3 4 5 6 7

(For instructor use only:)

1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
7	/20
Total	/100

1. For the differential equation $y'(t) = y(t) - t^2$ with $y(0) = 2$, verify that the exact solution is $y(t) = t^2 + 2t + 2$, so that $y(1) = 5$.

   (a) Use Euler's method to approximate $y(1)$ when $h = 0.5$.

   (b) Use the midpoint method to approximate $y(1)$ when $h = 1$.

   (c) Use the trapezoid method (the one-step Adams-Moulton method) to approximate $y(1)$ when $h = 1$. (Use the actual Adams-Moulton, not a predictor-corrector.)

2. The algorithm for Romberg integration is as follows.

```
h = b-a
R[1,1] = h/2 * (f(a)+f(b))
N = 1
for i from 2 to n do
   R[i,1] = 0
   x = a - 0.5*h
   % NOTE: N in the following loop is 2^(i-2)
   for k from 1 to N do
      x = x + h
      R[i,1] = R[i,1] + f(x)
   end do
   R[i,1] = (h*R[i,1]+R[i-1,1])/2
   M = 4
   for j from 2 to i do
      R[i,j] = R[i,j-1] + (R[i,j-1]-R[i-1,j-1])/(M-1)
      M = 4*M
   end do
   h = h/2
   N = 2*N
end do
```

Suppose each function evaluation $f(x)$ requires m multiplications/divisions.

Set up a formula that tells you the number of multiplications/divisions in total, in terms of m and n.

You do not need to evaluate or simplify this formula.

3. For each of the following characteristic polynomials arising from a multistep differential equation scheme, indicate whether the corresponding method is consistent or inconsistent, and whether it is stable or unstable. Explain your answers briefly.

   (a) $\lambda^3 - \frac{7}{6}\lambda^2 + \frac{1}{6} = (\lambda - 1)(\lambda - \frac{1}{2})(\lambda + \frac{1}{3})$

   (b) $\lambda^3 - \frac{3}{2}\lambda^2 + \frac{1}{2} = (\lambda - 1)(\lambda - 1)(\lambda + \frac{1}{2})$

   (c) $\lambda^3 + \frac{21}{20}\lambda^2 - \frac{1}{20} = (\lambda + 1)(\lambda - \frac{1}{5})(\lambda + \frac{1}{4})$

   (d) $\lambda^3 - \frac{13}{4}\lambda^2 + \frac{9}{4} = (\lambda - 1)(\lambda - 3)(\lambda + \frac{3}{4})$

4. Suppose we use the two-step difference scheme

$$w_{i+1} = -4w_i + 5w_{i-1} + 4hf(t_i, w_i) + 2hf(t_{i-1}, w_{i-1})$$

to approximate the equation $y'(t) = f(t, y(t))$.

(a) Compute the local truncation error of this method.

(b) Is the method convergent? Explain.

5. Define tridiagonal $n \times n$ matrices A_n as follows:

$$A_1 = \begin{pmatrix} 3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

etc.

(a) If $d_n = \det A_n$, show (for example, by expanding on the first row) that

$$d_n = 3d_{n-1} - 2d_{n-2},$$

for any $n \geq 3$.

(b) Find a general formula for d_n for any n by solving the difference equation.

6. (a) Give the Cholesky factorization $A = LL^t$ for the positive-definite matrix

$$A = \begin{pmatrix} 4 & -2 & 6 \\ -2 & 2 & -1 \\ 6 & -1 & 17 \end{pmatrix}.$$

You may use any method to find L, but check your work. (Hint: if you do it correctly, the entries of L are all integers.)

(b) Use your factorization to solve the system

$$\begin{pmatrix} 4 & -2 & 6 \\ -2 & 2 & -1 \\ 6 & -1 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \\ 8 \end{pmatrix}.$$

(Hint: again, the solutions should all be integers.)

7. For each of the following two situations, describe by name the method you would use to solve the problem numerically, if the goal is to minimize the amount of time taken by the program.

   Be specific: if the method is adaptive, specify how the error is estimated; if not, specify the step size.

   Also explain why you expect your method to be preferable to any other.

   (a) $\frac{dy}{dt} = g(t) + g(y)$ with $y(1) = 1$, and the function g is the solution of $g(x)^{g(x)} = 1$. All values of the solution must be known on the interval $[1, 3]$ to within an accuracy of 0.0001.

   (b) $\frac{dy}{dt} = y^2 - f(t)$ with $y(0) = 1$, where the function f is known to within 10^{-4}, but only at the values $f(0)$, $f(0.01)$, $f(0.02)$, etc. The solution is desired on the interval $[0, 1]$ to within the best possible accuracy.