

Math/AppM 4650, Spring 2009
Chain Project

The model is n particles in the plane \mathbb{R}^2 : $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$. They are joined by rigid links, each having length $1/n$, forming a chain of total length 1 (we can assume the units are chosen so this is true). The chain is suspended from the origin $(0, 0)$. So there are n constraint equations describing the links:

$$\begin{aligned} x_1^2 + y_1^2 &= 1/n^2 \\ (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 &= 1/n^2, \quad \text{for } 2 \leq i \leq n \end{aligned} \quad (1)$$

Ignoring friction, gravity, and self-collisions, the equations of motion are (using $x_0 = 0, y_0 = 0$):

$$\begin{aligned} \frac{d^2 x_i}{dt^2} &= n^2 \sigma_{i+1} (x_{i+1} - x_i) + n^2 \sigma_i (x_{i-1} - x_i), \quad \text{for } 1 \leq i \leq n-1 \\ \frac{d^2 x_n}{dt^2} &= n^2 \sigma_{n-1} (x_{n-1} - x_n), \\ \frac{d^2 y_i}{dt^2} &= n^2 \sigma_{i+1} (y_{i+1} - y_i) + n^2 \sigma_i (y_{i-1} - y_i), \quad \text{for } 1 \leq i \leq n-1 \\ \frac{d^2 y_n}{dt^2} &= n^2 \sigma_{n-1} (y_{n-1} - y_n). \end{aligned} \quad (2)$$

Here the numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ are the tensions in the chain, which are different in each link but always act in the direction of each link. Mathematically, they are the Lagrange multipliers corresponding to the constraints (1).

If we define angles $\theta_1, \dots, \theta_n$ by the formulas

$$(x_1, y_1) = \frac{1}{n} (\cos \theta_1, \sin \theta_1), \quad (x_i, y_i) = (x_{i-1}, y_{i-1}) + \frac{1}{n} (\cos \theta_i, \sin \theta_i), \quad \text{for } 2 \leq i \leq n, \quad (3)$$

then the evolution equations (2) simplify a bit to

$$\begin{aligned} \frac{d^2 \theta_1}{dt^2} &= n^2 \sigma_2 \sin(\theta_2 - \theta_1) \\ \frac{d^2 \theta_i}{dt^2} &= n^2 \sigma_{i+1} \sin(\theta_{i+1} - \theta_i) - n^2 \sigma_{i-1} \sin(\theta_i - \theta_{i-1}) \quad \text{for } 2 \leq i \leq n-1 \\ \frac{d^2 \theta_n}{dt^2} &= -n^2 \sigma_{n-1} \sin(\theta_n - \theta_{n-1}). \end{aligned} \quad (4)$$

We need to know what the tensions $\sigma_1, \dots, \sigma_n$ are. The trick for finding them is to differentiate the constraint equations (1) twice with respect to time, and then plug in the equations (2). This gives us the following equations:

$$\begin{aligned} \sigma_1 - \cos(\theta_2 - \theta_1) \sigma_2 &= \frac{\dot{\theta}_1^2}{n^2} \\ -\cos(\theta_{i+1} - \theta_i) \sigma_{i+1} + 2\sigma_i - \cos(\theta_i - \theta_{i-1}) \sigma_{i-1} &= \frac{\dot{\theta}_i^2}{n^2}, \quad \text{for } 2 \leq i \leq n-1 \\ 2\sigma_n - \cos(\theta_n - \theta_{n-1}) \sigma_{n-1} &= \frac{\dot{\theta}_n^2}{n^2}. \end{aligned} \quad (5)$$

Notice that this is a *tridiagonal* system for σ , so there is an efficient way to solve it given the θ_i and $\dot{\theta}_i$ for any fixed time.

Defining

$$\omega_i = \frac{d\theta_i}{dt}, \quad \text{for } 1 \leq i \leq n, \quad (6)$$

the n -dimensional second-order system (4) becomes the $2n$ -dimensional first-order system

$$\begin{aligned} \frac{d\theta_i}{dt} &= \omega_i, & \text{for } 1 \leq i \leq n, \\ \frac{d\omega_1}{dt} &= n^2 \sigma_2 \sin(\theta_2 - \theta_1) \\ \frac{d\omega_i}{dt} &= n^2 \sigma_{i+1} \sin(\theta_{i+1} - \theta_i) - n^2 \sigma_{i-1} \sin(\theta_i - \theta_{i-1}) & \text{for } 2 \leq i \leq n-1 \\ \frac{d\omega_n}{dt} &= -n^2 \sigma_{n-1} \sin(\theta_n - \theta_{n-1}), \end{aligned} \quad (7)$$

with σ determined by solving the $n \times n$ matrix

$$\begin{pmatrix} 1 & -\cos(\theta_2 - \theta_1) & 0 & \dots & 0 & 0 \\ -\cos(\theta_2 - \theta_1) & 2 & -\cos(\theta_3 - \theta_2) & \dots & 0 & 0 \\ 0 & -\cos(\theta_3 - \theta_2) & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\cos(\theta_n - \theta_{n-1}) & 2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} = \begin{pmatrix} \omega_1^2/n^2 \\ \omega_2^2/n^2 \\ \vdots \\ \omega_n^2/n^2 \end{pmatrix}. \quad (8)$$

Once we solve (numerically) for $\theta_1(t), \dots, \theta_n(t)$, we then use the equations (3) to get $x_1(t), \dots, x_n(t)$ and $y_1(t), \dots, y_n(t)$.

The main problem is to figure out what happens to this system as $n \rightarrow \infty$, when the “chain” approaches a “whip.” Although the equations certainly have solutions defined for all time for any particular n , because they are ordinary differential equations, in the limit they become a partial differential equation, and it is not known whether this partial differential equation actually has solutions defined for all time.

This problem is a simple analogy for the motion of an ideal, incompressible fluid: the length-preserving constraint (1) is analogous to the requirement that the fluid preserve volume, while the system (8) is analogous to the differential equation that determines the pressure of the incompressible fluid. The observation that the chain eventually “crinkles” is analogous to the development of turbulence in an incompressible fluid, and the question of whether solutions exist for all time, in fluid mechanics, is one of the most famous and long-standing problems in the field.

1. (Theoretical part) Explore the limit of the equations as $n \rightarrow \infty$: write $\theta_i(t) = \theta(t, \frac{i}{n})$ and $\sigma_i(t) = \sigma(t, \frac{i}{n})$, with $s = \frac{i}{n}$ for each fixed time t . Then $\theta(t, s)$ and $\sigma(t, s)$ are functions of both time t and the parameter s , with $0 \leq s \leq 1$. Specifically:

- (a) Show that equation (4) approaches the partial differential equation

$$\frac{\partial^2 \theta}{\partial t^2}(t, s) = \sigma(t, s) \frac{\partial^2 \theta}{\partial s^2}(t, s) + 2 \frac{\partial \sigma}{\partial s}(t, s) \frac{\partial \theta}{\partial s}(t, s).$$

Hint: for any fixed time, if $\theta_i = \theta(s)$ and $\theta_{i+1} = \theta(s + \frac{1}{n})$, then

$$\theta_{i+1} - \theta_i = \theta(s + \frac{1}{n}) - \theta(s) = \frac{1}{n}\theta'(s) + \frac{1}{2n^2}\theta''(s) + O(\frac{1}{n^3})$$

Then using the fact that $\sin u = u - \frac{u^3}{6} + O(u^5)$ for any $u \approx 0$, we know

$$\sin(\theta_{i+1} - \theta_i) = \frac{1}{n}\theta'(s) + \frac{1}{2n^2}\theta''(s) + O(\frac{1}{n^3}).$$

Use the same technique for the rest of the quantities in the equation; note you only need a few terms of the Taylor series, since the rest will go to zero as $n \rightarrow \infty$.

(b) Show the boundary condition at the fixed point must be

$$\sigma(t, 0) \frac{\partial \theta}{\partial s}(t, 0) = 0$$

to ensure that $\frac{\partial^2 \theta}{\partial t^2}(t, 0)$ is finite, by using the equation for $\frac{d^2 \theta_1}{dt^2}$.

(c) Show that the tension given by (5) approaches the solution of the differential equation

$$\frac{\partial^2 \sigma}{\partial s^2}(t, s) - \left(\frac{\partial \theta}{\partial s}(t, s) \right)^2 \sigma(t, s) = - \left(\frac{\partial \theta}{\partial t}(t, s) \right)^2.$$

(d) By looking at the equations for σ_1 and σ_n , show that the boundary conditions for σ must be

$$\frac{\partial \sigma}{\partial s}(t, 0) = 0 \quad \text{and} \quad \sigma(t, 1) = 0.$$

- Write a fast program to solve for σ at any time, given θ and ω at that time, using the algorithm in the book for tridiagonal systems. Check your program for $n = 3$ by hand to make sure it's giving correct values.
- Write a program to use Euler's method to solve the system, incorporating your program from problem 2 to get the σ values. (Note that the system is of the form

$$\frac{d}{dt} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \\ \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \\ n^2 \sigma_2 \sin(\theta_2 - \theta_1) \\ n^2 \sigma_3 \sin(\theta_3 - \theta_2) - n^2 \sigma_1 \sin(\theta_2 - \theta_1) \\ \vdots \\ -n^2 \sigma_{n-1} \sin(\theta_n - \theta_{n-1}) \end{pmatrix}.$$

Calling the left side a vector \mathbf{x} and the right side a vector function $\mathbf{f}(\mathbf{x})$, the equation is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, so that Euler's method for the system is just $\mathbf{x}_{m+1} = \mathbf{x}_m + h\mathbf{f}(\mathbf{x}_m)$.

Pick some initial condition for θ and $\dot{\theta}$. Choose a particular n (for example, $n = 20$), and figure out the smallest time step that gives a good approximation. Test it by repeating the entire calculation with half of that time step to make sure both curves are approximately the same over some time interval.

- Use a fourth-order method to get a better approximation with a smaller time step. (Since the function is very complicated, you want to avoid Taylor methods; the derivatives are a pain to compute. Instead use Runge-Kutta style methods or Adams-Bashforth predictor-corrector methods.) Since this is tricky to program, check your model against some reliable data from the Euler method above to make sure your program is working correctly.
- You need some initial conditions to get started. An easy one is $\theta_i(0) = 0$ for all i and $\dot{\theta}_i(0) = \frac{i}{n}$. A more interesting one is something like the following:

$$\theta_i = 2\pi \sin^2\left(\frac{\pi i}{2n}\right), \quad \omega_i = 2\frac{i^2}{n^2} - \frac{i}{n}.$$

This describes a string crossing itself and initially being pulled tight; as the loop disappears, the partial differential equation seems to break down. Exactly how this happens is the subject of current research. Try to come up with other interesting examples.

- For any of these simulations, one would ideally want animations on the same time-scale displaying the chains for different values of n . For example, starting with $n = 10$, then $n = 20$, then $n = 40$, etc., and seeing if there is any kind of convergence as n gets larger.

The main thing to plot is the actual chain in physical space: that is, compute the $x_i(t)$ and $y_i(t)$ using (3) and plot those; this way you can see if it behaves physically the way you'd expect.

Other things worth investigating:

- Does the tension σ always remain positive? Physically we might expect that each portion of the chain is always trying to stretch and being pulled back, but if σ were negative anywhere, it would mean the chain is compressing itself. In the limit of a whip, negative tension would be very surprising physically, but the equations don't rule it out.
 - What happens at the tip? The velocity ω_n can theoretically approach infinity as n approaches infinity; do you see it doing that? Is there any relation between crinkling of the chain and high velocity at the tip?
 - What do the graphs of θ , $\frac{\partial\theta}{\partial s}$, and $\frac{\partial\theta}{\partial t}$ look like? Do oscillations in those graphs diminish as n gets larger?
 - Of course, anything else you might want to explore would also be interesting.
- You should submit all of your programs, the derivations written out, the initial data you use for any simulations, and graphs and/or movies of your results (for various times, chain sizes, and initial conditions).

The most important thing is to try to discover at least one interesting phenomenon and demonstrate it convincingly through your numerical results. (This is how numerical research is done professionally.)