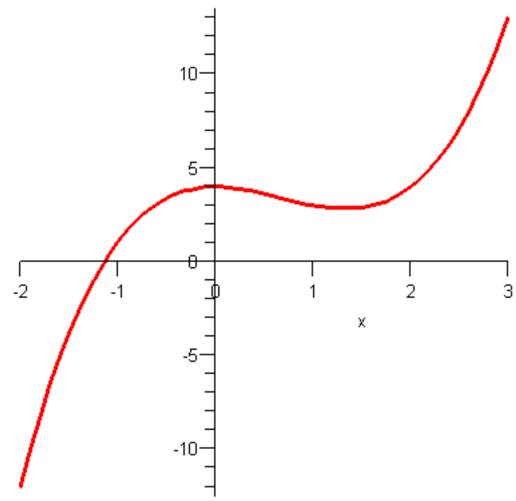
Math 4310 Homework #9 Solutions

1. For the function $f(x) = x^3 - 2x^2 + 4$ and the partition $P = \{-2, -1, 1, 3\}$ of $[-2, 3]$, compute $S^+(f, P)$ and $S^-(f, P)$ explicitly.

   Solution: Graph the function, as shown.



   Knowing that the critical points, where $f'(x) = 3x^2 - 4x = 0$, we know the local maximum occurs at $x = 0$ and the local minimum occurs at $x = \frac{4}{3}$. So

$$M_1 = \sup_{-2 \le x \le -1} f(x) = f(-1) = 1$$
$$M_2 = \sup_{-1 \le x \le 1} f(x) = f(0) = 4$$
$$M_3 = \sup_{1 \le x \le 3} f(x) = f(3) = 13$$
$$m_1 = \inf_{-2 \le x \le -1} f(x) = f(-2) = -12$$
$$m_2 = \inf_{-1 \le x \le 1} f(x) = f(-1) = 1$$
$$m_3 = \inf_{1 \le x \le 3} f(x) = f(\tfrac{4}{3}) = \frac{76}{27}.$$

   Thus the sums are

$$S^+(f, P) = M_1 + 2M_2 + 2M_3 = 35$$
$$S^-(f, P) = m_1 + 2m_2 + 2m_2 = -\frac{118}{27}.$$

2. Let f be a bounded function on $[a, b]$, and define the upper Riemann integral of f by

$$U(f) = \inf\{S^+(f, P) \mid P \text{ is a partition of } [a, b]\},$$

and the lower Riemann integral of f by

$$L(f) = \sup\{S^-(f,P) \,|\, P \text{ is a partition of } [a,b]\},$$

as in class.

(a) Prove that for every $\varepsilon > 0$, there is a partition P such that $U(f) > S^+(f,P) - \varepsilon$, and there is a partition P' such that $L(f) < S^-(f,P') + \varepsilon$.

Solution:

Since $U(f)$ is the infimum of all upper sums, we know that for every $\varepsilon > 0$, that $U(f) + \varepsilon$ is not a lower bound for the upper sums, or in other words there is an upper sum $S^+(f,P) < U(f) + \varepsilon$. This is exactly what we want. The same reasoning provides some lower sum with $S^-(f,P') > L(f) - \varepsilon$.

(b) Prove that f is Riemann integrable (i.e., $U(f) = L(f)$) if and only if, for every $\varepsilon > 0$, there is a partition P'' of $[a,b]$ such that

$$S^+(f,P'') - S^-(f,P'') < \varepsilon.$$

Solution:

First suppose $U(f) = L(f)$. Let $\varepsilon > 0$ be any number; then from part (a), we know there is a partition P such that $S^+(f,P) < U(f) + \frac{\varepsilon}{2}$. There is also a partition P' such that $S^-(f,P') > L(f) - \frac{\varepsilon}{2}$.

Let P'' be the union of the two partitions; then from a result in class, we know

$$S^+(f,P'') \le S^+(f,P) < U(f) + \frac{\varepsilon}{2}.$$

Similarly,

$$S^-(f,P'') \ge S^-(f,P') > L(f) - \frac{\varepsilon}{2}.$$

Now since $U(f) = L(f)$, then

$$S^+(f,P'') - S^-(f,P'') < U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2} = \varepsilon.$$

For the second part of the proof, suppose that for every $\varepsilon > 0$, we have a partition P'' such that $S^+(f,P'') - S^-(f,P'') < \varepsilon$. Then for any $\varepsilon > 0$, we have

$$U(f) - L(f) = S^+(f,P'') - S^-(f,P'') < \varepsilon.$$

Now the only way $U(f) - L(f)$ is less than every positive number is if $U(f) - L(f) \le 0$. But we already know that $L(f) \le U(f)$ for any function f, so we must have $L(f) = U(f)$. Hence f is integrable.

3. For the function $f(x) = x^2$ on $[0,b]$ and the equally-spaced partition $P_n = \{0, \frac{b}{n}, \frac{2b}{n}, \cdots, b\}$:

(a) Compute $S^+(f, P_n)$ and $S^-(f, P_n)$ explicitly. (Hint: look up a formula for $\sum_{k=1}^n k^2$ in terms of n.)

Solution:

The partition is $P_n = \{0, \frac{b}{n}, \frac{2b}{n}, \cdots, b\}$. Since the function is increasing on $[0, b]$, we know

$$\max_{x_{k_1} \le x \le x_k} f(x) = f(x_k) \qquad \text{and} \qquad \min_{x_{k-1} \le x \le x_k} f(x) = f(x_{k-1}).$$

Hence

$$\begin{aligned}
S^+(f, P_n) &= \sum_{k=1}^n f(x_k)(x_k - x_{k-1}) \\
&= \sum_{k=1}^n f\left(\tfrac{bk}{n}\right) \frac{b}{n} \\
&= \frac{b^3}{n^3} \sum_{k=1}^n k^2 \\
&= \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} \\
S^-(f, P_n) &= \sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1}) \\
&= \sum_{k=1}^n f\left(\tfrac{b(k-1)}{n}\right) \frac{b}{n} \\
&= \frac{b^3}{n^3} \sum_{k=0}^{n-1} k^2 \\
&= \frac{b^3}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - n^2 \right).
\end{aligned}$$

(b) Prove that for any $\varepsilon > 0$, there is an n such that $S^+(f, P_n) - S^-(f, P_n) < \varepsilon$. Why does this imply that $U(f) = L(f)$?

Solution:

It's clear that $S^+(f, P_n) - S^-(f, P_n) = \frac{b^3}{n}$. Hence for every $\varepsilon > 0$, choosing n to be an integer larger than $\frac{b^3}{\varepsilon}$ implies that $S^+(f, P_n) - S^-(f, P_n) < \varepsilon$.

This implies integrability by Problem 2: for every $\varepsilon > 0$ there is some partition P_n such that $S^+(f, P_n) - S^-(f, P_n) < \varepsilon$.

(c) Compute

$$\int_0^b x^2 \, dx = \lim_{n \to \infty} S^+(f, P_n)$$

explicitly.

Solution:

Easy enough.

$$\lim_{n\to\infty} S^+(f,P_n) = \frac{b^3}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{b^3}{3}.$$

4. (a) Prove that if f and g are bounded functions on $[a,b]$, then

$$U(f+g) \le U(f) + U(g).$$

Solutions:

Take any partition P of $[a,b]$. Then for $x \in [x_{k-1}, x_k]$ in P, we have

$$f(x)+g(x) \le \sup_{x_{k-1}\le x\le x_k} f(x) + \sup_{x_{k-1}\le x\le x_k} g(x).$$

Since this is true for all $x \in [x_{k-1}, x_k]$, we know

$$\sup_{x_{k-1}\le x\le x_k} \big(f(x)+g(x)\big) \le \sup_{x_{k-1}\le x\le x_k} f(x) + \sup_{x_{k-1}\le x\le x_k} g(x).$$

Hence adding these inequalities up, we see that for any partition P,

$$S^+(f+g,P) \le S^+(f,P) + S^+(g,P).$$

Now we want to be able to take the infimum of each of these separately. The trick is, letting $\varepsilon > 0$ be any number, we can find a partition P such that $S^+(f,P) < U(f) + \varepsilon/2$. (From problem 2(a).) Similarly there is a partition P' such that $S^+(g,P') < U(g) + \varepsilon/2$. Let P'' be the union of P and P'. Then

$$S^+(f+g,P'') \le S^+(f,P'')+S^+(g,P'') < U(f)+\frac{\varepsilon}{2}+U(g)+\frac{\varepsilon}{2} = U(f)+U(g)+\varepsilon.$$

Since $U(f+g) \le S^+(f+g,P'')$ (because $U(f+g)$ is a lower bound for the upper sums), we know

$$U(f+g) < U(f)+U(g)+\varepsilon.$$

This is true for every ε, and hence

$$U(f+g) \le U(f) + U(g).$$

(b) Give an *explicit* example of two functions f and g and an interval $[a,b]$ such that $U(f+g) < U(f)+U(g)$ on $[a,b]$. (Hint: think about the nonintegrable example from class.)

Solution:

Let $f\colon [0,1] \to \mathbb{R}$ be the nonintegrable function given by

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}.$$

Define $g(x) = 1 - f(x)$.

Then obviously $(f+g)(x) = 1$ for all x, which is integrable, and $U(f+g) = 1$.

On the other hand, for any partition P, there are rational numbers between x_{k-1} and x_k. This implies $S^+(f,P) = 1$. This is true for every partition P, and hence $U(f) = 1$. Similarly, there are *irrational* numbers between x_{k-1} and x_k for every partition, which implies $S^+(g,P) = 1$. So $U(g) = 1$.

Thus $U(f+g) = 1 < 2 = U(f) + U(g)$.

5. Prove carefully that if f and g are both Riemann integrable on $[a,b]$, then so is $f+g$. Furthermore,

$$\int_a^b \big(f(x) + g(x)\big) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(Hint: to prove $f+g$ is integrable, use $U(f+g) < U(f) + U(g)$ from the previous problem. Find a similar inequality for $L(f+g)$.)

Solution:

We know that $U(f+g) \le U(f) + U(g)$. By the same reasoning as in the previous problem, we see $S^-(f+g,P) \ge S^-(f,P) + S^-(g,P)$ for every partition; thus, using the same techniques, we get $L(f+g) \ge L(f) + L(g)$.

Since $L(f) = U(f)$ and $L(g) = U(g)$, we see that $L(f+g) \ge L(f) + L(g)$ and also that $U(f+g) \le U(f) + U(g)$. Combining these we get

$$L(f+g) \ge L(f) + L(g) = U(f) + U(g) \ge U(f+g),$$

so that $L(f+g) \ge U(f+g)$. But we always know that $L(f+g) \le U(f+g)$. The only way both of these are true is if $L(f+g) = U(f+g)$, i.e., if $f+g$ is integrable.

In this case, $L(f+g) = L(f) + L(g)$ and $U(f+g) = U(f) + U(g)$, so that

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x).$$

6. Prove carefully that if $a < b < c$, and if f is integrable on $[a,c]$, then f is also integrable on $[a,b]$ and on $[b,c]$. Furthermore,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

(Hint: For any partition P of $[a,c]$ which contains b, there are partitions P' of $[a,b]$ and P'' of $[b,c]$ such that $P = P' \cup P''$. Furthermore, show that $S^+(f,P) = S^+(f,P') + S^+(f,P'')$ and $S^-(f,P) = S^-(f,P') + S^-(f,P'')$. What can you do with this information?)

Solution:

Let $\varepsilon > 0$ be any number, and choose a partition P of $[a,c]$ such that $S^+(f,P) - S^-(f,P) < \varepsilon$. (We can do this because f is integrable on $[a,c]$, by problem 2b.) Add b

to P to get a new partition P_o; since $S^+(f, P_o) \le S^+(f, P)$ and $S^-(f, P_o) \ge S^-(f, P)$, we know

$$S^+(f, P_o) - S^-(f, P_o) \le S^+(f, P) - S^-(f, P) < \varepsilon.$$

Now divide up $P_o = \{a, x_1, \cdots, x_{k-1}, b, x_{k+1}, \cdots, x_{n-1}, c\}$ into two partitions

$$P' = \{a, x_1, \cdots, x_{k-1}, b\} \qquad \text{and} \qquad P'' = \{b, x_{k+1}, \cdots, x_{n-1}, c\}.$$

Then obviously $S^+(f, P_o) = S^+(f, P') + S^+(f, P'')$ and $S^-(f, P_o) = S^-(f, P') + S^-(f, P'')$. So

$$\left[S^+(f, P') - S^-(f, P')\right] + \left[S^+(f, P'') - S^-(f, P'')\right] = S^+(f, P_o) + S^-(f, P_o) < \varepsilon.$$

Now both terms in square brackets are nonnegative, and so the only way the sum of them can be less than ε is if each term is less than ε. Thus

$$S^+(f, P') - S^-(f, P') < \varepsilon \qquad \text{and} \qquad S^+(f, P'') - S^-(f, P'') < \varepsilon.$$

For any ε, we found such an ε, and that means (by problem 2b) that f is integrable on $[a, b]$ and on $[b, c]$.

Furthermore, the inequality $S^+(f, P') + S^+(f, P'') \le S^+(f, P)$ implies that $U_{[a,b]}(f) + U_{[b,c]}(f) \le S^+(f, P)$ for all partitions P. Thus $U_{[a,b]}(f) + U_{[b,c]}(f) \le U_{[a,c]}(f)$. Similarly $L_{[a,b]}(f) + L_{[b,c]}(f) \ge L_{[a,c]}(f)$. Now

$$U_{[a,b]}(f) = L_{[a,b]}(f), \qquad U_{[b,c]}(f) = L_{[b,c]}(f), \qquad \text{and } U_{[a,c]}(f) = L_{[a,c]}(f).$$

So we have

$$U_{[a,b]}(f) + U_{[b,c]}(f) \ge U_{[a,c]}(f) \qquad \text{and} \qquad U_{[a,b]}(f) + U_{[b,c]}(f) \le U_{[a,c]}(f).$$

Hence they are equal, and we have

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx.$$

7. (a) Using the result of the previous problem, prove that if f is a bounded, Riemann integrable function on $[a, b]$ and we define

$$F(x) = \int_a^x f(t)\, dt$$

for all $x \in [a, b]$, then F is a continuous function. (Hint: What is $F(x_o + h) - F(x_o)$?)

Solution:

We know that f is integrable on every subinterval of $[a, b]$ by the previous problem. Furthermore, if $h > 0$ we know

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$

Now since f is bounded, there is a number M such that $f(t) \le M$ for all $t \in [a, b]$. So the integral over $[x, x+h]$ is less than the upper Riemann sum with the two-point partition $P = \{x, x+h\}$:

$$\int_x^{x+h} f(t)\,dt \le h \sup_{[x,x+h]} f(t) \le hM.$$

Hence

$$F(x+h) - F(x) \le hM.$$

Similarly there exists m such that $f(t) \ge m$ for all $t \in [a, b]$, which implies (using the lower Riemann sum) that

$$F(x+h) - F(x) \ge hm.$$

Combining, we get

$$-h|m| \le F(x+h) - F(x) \le h|M|.$$

Now we can conclude that

$$|F(x+h) - F(x)| \le |h| \min\{|m|, |M|\}.$$

Similarly we can do the same trick when $h < 0$: we have

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^{x+h} f(t)\,dt - \int_{x+h}^x f(t)\,dt \\ &= -\int_{x+h}^x f(t)\,dt. \end{aligned}$$

We have

$$mh \le \int_{x+h}^x f(t)\,dt \le Mh,$$

so that

$$-Mh \le F(x+h) - F(x) \le -mh.$$

Again we conclude that

$$-h|M| \le F(x+h) - F(x) \le h|m|,$$

and from this we derive

$$|F(x+h)-F(x)| \leq |h| \min\{|m|, |M|\}.$$

Now let $\varepsilon > 0$, and set $\delta > 0$ to be $\delta = \frac{\varepsilon}{\min\{m,M\}}$. Then $|h| < \delta$ implies that

$$|F(x+h)-F(x)| \leq |h| \min\{|m|, |M|\} < \varepsilon.$$

So F is continuous (in fact, uniformly continuous).

(b) Suppose

$$f(x) = \begin{cases} 1 & x \geq 1 \\ 0 & x < 1. \end{cases}$$

Compute $F(x)$ explicitly and verify that F is continuous but not differentiable at $x_o = 1$.

Solution:

If we define

$$F(x) = \int_0^x f(t)\,dt,$$

we get

$$F(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ (x-1) & 1 \leq x \end{cases}.$$

Clearly F is continuous since

$$\lim_{x\to 1^-} F(x) = 0 = \lim_{x \to 1^+} F(x).$$

However F is not differentiable since

$$\lim_{x\to 1^-} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^-} \frac{0-0}{x-1} = 0$$

but

$$\lim_{x\to 1^+} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1.$$