Math 4310 Homework #8 Solutions

1. Textbook problem, 5.2.4: 3.

   Is the converse of the mean value theorem true, in the sense that if f is continuous on $[a,b]$ and differentiable on (a,b), given a point x_o in (a,b) must there exist points x_1, x_2 in $[a,b]$ such that
   $$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_o)?$$

   Solution:

   Let x_o be given, and define $L = f'(x_o)$. The easiest way to understand this is by considering the new function
   $$g(x) = f(x) - Lx.$$
   Then $g'(x_o) = 0$ and $f(x_2) - f(x_1) = g(x_2) - g(x_1) + L(x_2 - x_1)$, so
   $$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_o) \Leftrightarrow \frac{g(x_2)-g(x_1)}{x_2-x_1} + L = L \Leftrightarrow g(x_2) = g(x_1).$$

   So we are basically asking whether, if $g'(x_o) = 0$ for some x_o, there must be distinct points x_1 and x_2 with $g(x_1) = g(x_2)$. This is false: a simple counterexample is $g(x) = x^3$ on the interval $[-1,1]$. Then $g'(0) = 0$, but g is one-to-one because $x_1^3 = x_2^3$ implies $x_1 = x_2$.

2. Textbook problem, 5.2.4: 7.

   Draw the graph of a function that has a local maximum that is not a strict local maximum but is not constant on an interval.

   Solution:

   To have a local maximum that is not a strict local maximum, we must have $f(x_o) \geq f(x)$ for all x in some open interval, but such that there is an x arbitrarily close to x_o with $f(x) = f(x_o)$. To get this without having f actually constant on an interval, we need rapid oscillations as happen with $\sin(1/x)$. Figure 1 shows an example.

3. Textbook problem, 5.3.4: 1.

   Define
   $$x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

   Prove that $f(x) = x_+^k$ is continuously differentiable if k is an integer greater than one.

   Solution:

   We know that if $x_o > 0$, then $f(x) = x^k$ for all x near x_o, and hence $f'(x_o) = kx_o^{k-1}$. Furthermore if $x_o < 0$, then $f(x) = 0$ for all x near x_o, and hence $f'(x_o) = 0$. So f is continuously differentiable on $(0,\infty)$ and on $(-\infty,0)$; to get continuously differentiable on all of $\mathbb{R}$, we just need to show $\lim_{x\to 0} f'(x) = f'(0)$.

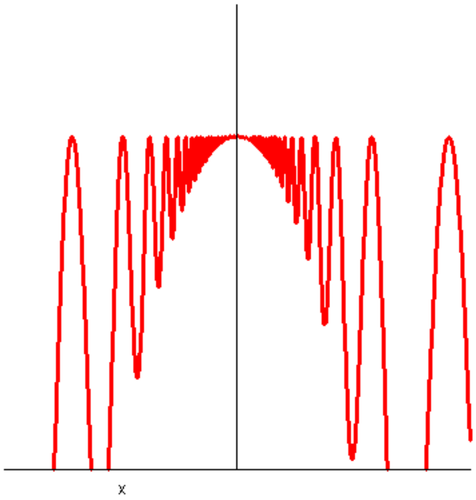

Figure 1: A function which is not constant on any interval, with a local maximum that is not a strict local maximum.

So first of all, $f'(0) = \lim_{x\to 0} \frac{f(x)}{x}$, since $f(0) = 0$. Secondly, if $x > 0$ then $\frac{f(x)}{x} = x^{k-1}$, while if $x < 0$ then $\frac{f(x)}{x} = 0$; either way, since $k > 1$ we can choose for any $\varepsilon > 0$ a number $\delta = \varepsilon^{1/(k-1)}$ so that $|x| < \delta$ implies $|\frac{f(x)}{x}| \le |x^{k-1}| < \delta^{k-1} = \varepsilon$. So $f'(0) = 0$.

Finally we know that $f'(x) = kx^{k-1}$ or $f'(x) = 0$ for all $x \neq 0$, and hence $|f'(x) - f'(0)| \le k|x|^{k-1}$. So for any $\varepsilon > 0$, we can choose $\delta = (\varepsilon/k)^{1/(k-1)}$; then $|x| < \delta$ implies

$$|f'(x) - f'(0)| \le k|x|^{k-1} < k\delta^{k-1} = \varepsilon.$$

Hence $\lim_{x\to 0} f'(x) = f'(0)$. So f is continuously differentiable.

4. Textbook problem, 5.3.4: 2.

   Show that $(x-a)_+^2(b-x)_+^2$ is a continuously differentiable function that is non-zero exactly on the interval (a, b).

   Solution:

   We can write $h(x) = (x-a)_+^2(b-x)_+^2$ as a product of simpler functions: $h(x) = f(x)g(x)$ where $f(x) = (x-a)_+^2$ and $g(x) = (b-x)_+^2$. By the product rule, h is differentiable if both f and g are differentiable. Furthermore since $h' = fg' + f'g$, if f and g are continuously differentiable, then f', g', f, and g are all continuous functions, so h' is also a continuous function. So it's enough to prove that f and g are both continuously differentiable.

   Now f is continuously differentiable because we can write it as a composition of functions, $f(x) = q\big(r(x)\big)$, where $r(x) = x - a$ and $q(y) = y_+^2$. In the previous problem, we showed that q is continuously differentiable, and obviously r is continuously differentiable, so by the chain rule f is also differentiable, and since $f'(x) = q'\big(r(x)\big)r'(x)$, we have a product of a composition of continuous functions, so f' is continuous. The proof that g' is continuous is exactly the same.

Hence h' exists and is continuous.

5. Textbook problem, 5.3.4: 4.

   Prove that if f is any continuous one-to-one function on an interval (a, b), then either f is strictly increasing or strictly decreasing.

   Solution:

   Let x, y, and z be any three distinct points with z between x and y. We want to show first that $f(z)$ must be between $f(x)$ and $f(y)$. Since f is one-to-one, we know $f(z) \neq f(x)$ and $f(z) \neq f(y)$. So there are three cases: either $f(z)$ is between $f(x)$ and $f(y)$, or $f(z)$ is larger than both, or $f(z)$ is smaller than both. In the first case, we have what we want.

   In the second case, $f(z)$ is larger than both $f(x)$ and $f(y)$. Now since $f(x) \neq f(y)$, one is larger than the other; suppose that $f(x)$ is larger than $f(y)$. Then $f(x)$ is between $f(y)$ and $f(z)$, so that by the Intermediate Value Theorem, there is some w between y and z with $f(w) = f(x)$. But $w \neq x$, since w is on the other side of z from x. This contradicts the fact that f is one-to-one. The third case (where $f(z)$ is smaller than both) also leads to a contradiction.

   Hence for any three points x, y, and z, the fact that z is between x and y implies $f(z)$ is between $f(x)$ and $f(y)$.

   So now we can prove that for any closed interval $[c, d]$ in (a, b), f must be strictly monotone on $[c, d]$. Suppose $f(c) < f(d)$; then first if $x \in [c, d]$, then $f(c) < f(x) < f(d)$ as before by the betweenness property. Similarly if $y < x$ then $f(c) < f(y) < f(x)$, and if $y > x$, then $f(x) < f(y) < f(d)$, by the betweenness property. So $f(c) < f(d)$ implies f is strictly increasing on all of $[c, d]$. Similarly $f(c) > f(d)$ implies f is strictly decreasing on all of $[c, d]$.

   Now if f is strictly increasing on *any* of these closed intervals, then it must be strictly increasing on every interval containing it, and hence on the union of all of them (which is all of (a, b)). Similarly if f is strictly decreasing on any closed interval, it must be strictly decreasing on every interval, and hence on (a, b).

6. Textbook problem, 5.3.4: 5.

   If f is differentiable on (a, b) and $f'(x) \neq 0$ for all x in the interval, prove that either $f'(x) > 0$ or $f'(x) < 0$ on the entire interval.

   Solution: Suppose that f is differentiable on (a, b) and $f'(x) \neq 0$. Assume (to get a contradiction) that f' is neither always positive nor always negative. Then there is some $x_1 \in (a, b)$ with $f'(x_1) > 0$ and some $x_2 \in (a, b)$ with $f'(x_2) < 0$. By the intermediate value theorem for derivatives, there must be a point x_o between x_1 and x_2 such that $f'(x_o) = 0$. This is a contradiction, and thus our assumption must be false.

7. Suppose a function $f(x)$ is twice-differentiable on (a, b) (i.e., f' exists on (a, b) and is differentiable everywhere; notice f and f' must be continuous, but f'' may not be continuous).

If $f'(x) > 0$ for all x, then the inverse function theorem tells us that f^{-1} exists on an interval (c, d) and is differentiable everywhere.

(a) If $(f^{-1})''(y_o)$ exists for some y_o, what must it be in terms of f, f', and f''? (Hint: differentiate the formula $f\big(f^{-1}(y)\big) = y$ twice.)

Solution:

Assuming $(f^{-1})''(y_o)$ exists, we can use the chain rule to compute it. For all y near y_o, we have

$$f\big(f^{-1}(y)\big) = y,$$

and therefore the derivative of both functions with respect to y is

$$f'\big(f^{-1}(y)\big)(f^{-1})'(y) = 1$$

for all y near y_o.

Differentiating again with respect to y, we get (using both the product and chain rules)

$$f''\big(f^{-1}(y_o)\big)\big[(f^{-1})'(y_o)\big]^2 + f'\big(f^{-1}(y)\big)(f^{-1})''(y_o) = 0.$$

Hence solving for $(f^{-1})''(y_o)$, we get

$$(f^{-1})''(y_o) = -\frac{f''\big(f^{-1}(y_o)\big)}{\big[f'\big(f^{-1}(y_o)\big)\big]^3}.$$

(b) Prove that $(f^{-1})''(y)$ exists at every $y \in (c, d)$. (Hint: write out the definition, simplify, and use a multiply-and-divide trick.)

Solution:

From the definition of the derivative, we have

$$\begin{aligned}
(f^{-1})''(y_o) &= \lim_{y \to y_o} \frac{(f^{-1})'(y) - (f^{-1})'(y_o)}{y - y_o} \\
&= \lim_{y \to y_o} \frac{\frac{1}{f'\left(f^{-1}(y)\right)} - \frac{1}{f'\left(f^{-1}(y_o)\right)}}{y - y_o} \\
&= \lim_{y \to y_o} \frac{f'\big(f^{-1}(y_o)\big) - f'\big(f^{-1}(y)\big)}{f'\big(f^{-1}(y)\big) f'\big(f^{-1}(y_o)\big)[y - y_o]} \\
&= -\lim_{y \to y_o} \frac{f'\big(f^{-1}(y)\big) - f'\big(f^{-1}(y_o)\big)}{f^{-1}(y) - f^{-1}(y_o)} \, \frac{f^{-1}(y) - f^{-1}(y_o)}{y - y_o} \, \frac{1}{f'\big(f^{-1}(y)\big)\, f'\big(f^{-1}(y_o)\big)} \\
&= -\left(\lim_{y \to y_o} \frac{f'\big(f^{-1}(y)\big) - f'\big(f^{-1}(y_o)\big)}{f^{-1}(y) - f^{-1}(y_o)}\right) \left(\lim_{y \to y_o} \frac{f^{-1}(y) - f^{-1}(y_o)}{y - y_o}\right) \\
&\qquad \cdot \left(\lim_{y \to y_o} \frac{1}{f'\big(f^{-1}(y)\big)\, f'\big(f^{-1}(y_o)\big)}\right) \\
&= -\left(\lim_{y \to y_o} \frac{f'\big(f^{-1}(y)\big) - f'\big(f^{-1}(y_o)\big)}{f^{-1}(y) - f^{-1}(y_o)}\right) (f^{-1})'(y_o) \, \frac{1}{f'\big(f^{-1}(y_o)\big)^2},
\end{aligned}$$

where we used the fact that f^{-1} is differentiable at y_o as well as the fact that f' is continuous.

Now to compute $\lim_{y \to y_o} \frac{f'\left(f^{-1}(y)\right) - f'\left(f^{-1}(y_o)\right)}{f^{-1}(y) - f^{-1}(y_o)}$, we rename (for simplicity) $x = f^{-1}(y)$ and $x_o = f^{-1}(y_o)$. Then by continuity of f^{-1}, the limit $y \to y_o$ implies $f^{-1}(y) \to f^{-1}(y_o)$ which implies $x \to x_o$. So

$$\lim_{f^{-1}(y) \to f^{-1}(y_o)} \frac{f'\big(f^{-1}(y)\big) - f'\big(f^{-1}(y_o)\big)}{f^{-1}(y) - f^{-1}(y_o)} = \lim_{x \to x_o} \frac{f'(x) - f'(x_o)}{x - x_o} = f''(x_o).$$

Thus finally we get

$$(f^{-1})''(y_o) = -\frac{f''\big(f^{-1}(y_o)\big)}{\big[f'\big(f^{-1}(y_o)\big)\big]^3}.$$

(The reason this proof works is that we already know $f^{-1}(y) \neq f^{-1}(y_o)$, since f^{-1} is strictly increasing. Hence the multiply and divide trick works here, although it doesn't work for the proof of the ordinary chain rule.)