Math 4310 Homework #7 Solutions

1. (a) Prove that $f(x) = x^3$ is differentiable at $x_o = 1$, using the definition of the derivative.

Solution:

First of all, we compute

$$\frac{f(x)-f(1)}{x-1} = \frac{x^3-1}{x-1} = x^2+x+1.$$

This appears to be approaching $f'(1) = 3$. To verify this from the definition, we try to find, for any $\varepsilon > 0$, a $\delta > 0$ such that for all x with $|x-1| < \delta$, we have

$$\left|\frac{f(x)-f(1)}{x-1} - 3\right| < \varepsilon.$$

So let $\varepsilon > 0$ be any number. Then

$$|x^2+x+1-3| = |x^2+x-2| = |x+2||x-1|.$$

We know that $|x-1|$ can be made as small as desired, so we just need $|x+2|$ to not be too large. When $x = 1$ we know $|x+2| = 3$; therefore since it's approaching 3, it must eventually be less than 4 (or any number larger than 3). Obviously $|x+2| < 4$ if $|x-1| < 1$. Thus as long as $|x-1| < 1$, we know

$$\left|\frac{f(x)-f(1)}{x-1} - 3\right| < 4|x-1|.$$

We want the right hand side to be less than ε, so choose δ to be less than 1 and also less than $\varepsilon/4$. Then $|x-1| < \delta$ implies

$$\left|\frac{f(x)-f(1)}{x-1} - 3\right| < \varepsilon.$$

(b) Prove from the definition that $f(x) = \sqrt{x}$ is *not* differentiable at $x_o = 0$.

Solution: The definition of 'differentiable' is, "There exists an L such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every x with $|x - x_o| < \delta$, we have $|\frac{f(x)-f(x_o)}{x-x_o} - L| < \varepsilon$."

So the negation is, "For every L, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there is some x with $|x-x_o| < \delta$ and $|\frac{f(x)-f(x_o)}{x-x_o} - L| \geq \varepsilon$."

So let L be any candidate for the derivative. Choose $\varepsilon = 1$. Then we know

$$\begin{aligned}\left|\frac{f(x)-f(x_o)}{x-x_o} - L\right| &= \left|\frac{\sqrt{x}-0}{x-0} - L\right| \\ &= |1/\sqrt{x} - L| \\ &\geq \frac{1}{\sqrt{x}} - |L|.\end{aligned}$$

So let $\delta > 0$ be any number. Choose x such that $\frac{1}{\sqrt{x}} \geq |L| + 1$ and also $|x| < \delta$; $x = \min\{\delta/2, \frac{1}{(|L|+1)^2}\}$ will work. Then $|x| < \delta$ and also

$$\left| \frac{f(x) - f(0)}{x - 0} - L \right| \geq \frac{1}{\sqrt{x}} - |L| \geq 1 = \varepsilon.$$

2. Textbook problem, 5.1.3: 1.

   Show that $f(x) = O(|x - x_o|^2)$ as $x \to x_o$ implies $f(x) = o(|x - x_o|)$ as $x \to x_o$, but give an example to show that the converse is not true.

   Solution:

   By definition, we know $f(x) = O|x - x_o|^2)$ implies there is some $M > 0$ and $\delta_1 > 0$ such that whenever $|x - x_o| < \delta_1$¡ we have $|f(x)| \leq M|x - x_o|^2$. We want to prove that $f(x) = o(|x - x_o|)$, i.e., that for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x)| \leq \varepsilon|x - x_o|$.

   Now the idea here is that we can split up $M|x - x_o|^2$ into $M|x - x_o|$ and $|x - x_o|$; the first term can be made smaller than ε, while the second term just carries over. So choose $\delta_2 = \frac{\varepsilon}{M}$, and let $\delta = \min\{\delta_1, \delta_2\}$. Then $|x - x_o| < \delta$ implies $|x - x_o| < \delta_1$ so that $|f(x)| \leq M|x - x_o|^2$; it also implies that $|x - x_o| < \frac{\varepsilon}{M}$, so that

   $$|f(x)| \leq \varepsilon|x - x_o|.$$

   Hence we are done.

   Now for the counterexample, we want to find a function $f(x)$ such that $\frac{|f(x)|}{|x - x_o|}$ approaches 0, but $\frac{|f(x)|}{|x - x_o|^2}$ is unbounded (i.e., approaches infinity). Essentially we want something between $|x - x_o|$ and $|x - x_o|^2$; the function $f(x) = |x - x_o|^{3/2}$ works. It's obvious that this satisfies both conditions.

3. Textbook problem, 5.1.3: 2.

   Show that $f(x) = O\big(|x - x_o|^k\big)$ and $g(x) = O\big(|x - x_o|^k\big)$ imply $(f + g)(x) = O\big(|x - x_o|^k\big)$. Is the same true of "little oh"?

   Solution:

   By definition, there is an M_1 and a δ_1 such that $|x - x_o| < \delta_1$ implies $|f(x)| \leq M_1|x - x_o|^k$, and there is an M_2 and a δ_2 such that $|x - x_o| < \delta_2$ implies $|g(x)| \leq M_2|x - x_o|^k$. Now let $\delta = \min\{\delta_1, \delta_2\}$, and let $M = M_1 + M_2$.

   Then $|x - x_o| < \delta$ implies that both conditions are met, so that

   $$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq M_1|x - x_o|^k + M_2|x - x_o|^k \\ &= (M_1 + M_2)|x - x_o|^k \\ &= M|x - x_o|^k. \end{aligned}$$

   Hence $(f + g)(x) = O(|x - x_o|^k)$.

The same is true for "little oh," since the conditions $f(x) = o(|x - x_o|^k)$ and $g(x) = o(|x - x_o|^k)$ imply together that $\lim_{x\to x_o} \frac{f(x)}{|x-x_o|^k} = 0$ and $\lim_{x\to x_o} \frac{g(x)}{|x-x_o|^k} = 0$. Hence the limit of the sum is the sum of the limits, and $(f + g)(x) = o(|x - x_o|^k)$. (We could prove this directly from the definition by using an $\varepsilon/2$ argument.)

4. Suppose we define a function f to be *symmetrically differentiable at x_o* if

$$f_s'(x_o) = \lim_{h\to 0} \frac{f(x_o + h) - f(x_o - h)}{2h}$$

exists.

(a) Prove that $f(x) = |x|$ is symmetrically differentiable at $x_o = 0$ but not differentiable at $x_o = 0$.

Solution:

$$f_s'(0) = \lim_{h\to 0} \frac{f(h) - f(-h)}{2h} = \lim_{h\to 0} \frac{|h| - |-h|}{2h} = \lim_{h\to 0} 0 = 0.$$

So the symmetric derivative exists.

However, we know that the ordinary derivative does not exist, since when $h > 0$, we know $\frac{f(h)-f(0)}{h} = \frac{|h|}{h} = 1$, while when $h < 0$, we know $\frac{f(h)-f(0)}{h} = \frac{|h|}{h} = -1$. Hence we can find values of h, arbitrarily close to zero, such that the difference quotient is 1 or -1; hence it cannot have a limit.

(b) Prove from the definitions that if f is differentiable at x_o, then f is symmetrically differentiable at x_o, and $f_s'(x_o) = f'(x_o)$.

Solution:

The key is to get the symmetric derivative in terms of the ordinary derivative, by adding and subtracting $f(x_o)$ in the definition.

Suppose f is differentiable at x_o. Then

$$\begin{aligned}
f_s'(x_o) &= \lim_{h\to 0} \frac{f(x_o + h) - f(x_o - h)}{2h} \\
&= \lim_{h\to 0} \frac{f(x_o + h) - f(x_o) + f(x_o) - f(x_o - h)}{2h} \\
&= \lim_{h\to 0} \frac{f(x_o + h) - f(x_o)}{2h} + \lim_{h\to 0} \frac{f(x_o) - f(x_o - h)}{2h} \\
&= \frac{1}{2} f'(x_o) + \lim_{h\to 0} \frac{f(x_o) - f(x_o - h)}{2h}.
\end{aligned}$$

To simplify the second term, observe that it looks like the definition of the derivative, just with a $-h$ instead of h. So let $j = -h$; then $h \to 0$ iff $j \to 0$ since

$|h| = |j|$. So with $h = -j$, we get

$$\begin{aligned} f_s'(x_o) &= \frac{1}{2}f'(x_o) + \lim_{j\to 0}\frac{f(x_o) - f(x_o + j)}{-2j} \\ &= \frac{1}{2}f'(x_o) + \lim_{j\to 0}\frac{f(x_o + j) - f(x_o)}{2j} \\ &= \frac{1}{2}f'(x_o) + \frac{1}{2}f'(x_o) = f'(x_o). \end{aligned}$$

5. Textbook problem, 5.2.4: 1.

   Let f and g be continuous functions on $[a, b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point x_o in (a, b) such that

   $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_o)}{g'(x_o)}.$$

   (Hint: apply the mean value theorem to the function

   $$\big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x).)$$

   This is sometimes called the *second mean value theorem.*

   Solution:

   As suggested in the hint, let $h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x)$. Then h is also differentiable on (a, b) and continuous on $[a, b]$, so h satisfies the hypotheses of the mean value theorem. Hence there exists x_o such that

   $$\frac{h(b) - h(a)}{b - a} = h'(x_o).$$

   Now we just have to translate this into statements about f and g. We know

   $$\begin{aligned} h(a) &= \big(f(b) - f(a)\big)g(a) - \big(g(b) - g(a)\big)f(a) = f(b)g(a) - g(b)f(a) \\ h(b) &= \big(f(b) - f(a)\big)g(b) - \big(g(b) - g(a)\big)f(b) = f(b)g(a) - f(a)g(b) \\ h'(x_o) &= \big(f(b) - f(a)\big)g'(x_o) - \big(g(b) - g(a)\big)f'(x_o). \end{aligned}$$

   So $h(b) = h(a)$, and hence $h(b) - h(a) = 0$. So also $h'(x_o) = 0$. Thus

   $$\big(f(b) - f(a)\big)g'(x_o) = \big(g(b) - g(a)\big)f'(x_o),$$

   which implies

   $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_o)}{g'(x_o)}.$$

6. Textbook problem, 5.2.4: 2.

   If f is a function satisfying

   $$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all x and y and some fixed M and $\alpha > 1$, prove that f is constant. (Hint: what is f'?)

Solution:

Let x_o be any fixed number; we want to prove $f'(x_o)$ exists and is equal to zero. So we compute

$$\left|\frac{f(x) - f(x_o)}{x - x_o}\right| \leq \frac{|x - x_o|^\alpha}{|x - x_o|} = |x - x_o|^{\alpha-1}.$$

Let $\varepsilon > 0$ be any number, and choose $\delta = \varepsilon^{1/(\alpha-1)}$. Then $\delta^{\alpha-1} = \varepsilon$, so that $|x - x_o| < \delta$ implies $|x - x_o|^{\alpha-1} < \delta^{\alpha-1} = \varepsilon$. (Here we used the fact that $\alpha > 1$, since we needed the exponent to be *positive* to preserve the inequality.) Hence $|x - x_o| < \delta$ implies

$$\left|\frac{f(x) - f(x_o)}{x - x_o}\right| < \varepsilon.$$

Hence $f'(x_o) = 0$.

Since $f'(x_o) = 0$ for all x_o, we know that f must be constant by Theorem 5.2.2c.

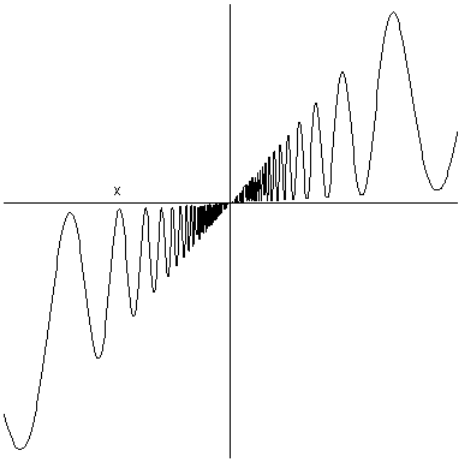
7. Textbook problem, 5.2.4: 5.

Draw a picture of the graph of a function that is strictly increasing at a point but is not even monotone increasing in a neighborhood of that point.

Solution:

Let's say the point is $x = 0$ and that $f(0) = 0$, to make things easy. For strictly increasing at $x = 0$, we want $f(x) > 0$ whenever $x > 0$, and $f(x) < 0$ whenever $x < 0$. So we just need the entire right side of the function's graph to be strictly above the x-axis, and the entire left side of the function's graph to be strictly below the x-axis.

To get a function that is not even monotone increasing in a neighborhood of 0, we want the function to oscillate a lot near 0, so we want something like a $\sin(1/x)$-type of oscillation. The following is the basic idea.



8. Suppose f is differentiable everywhere on (a, b), and that there is a number M with $|f'(x)| \leq M$ for all $x \in (a, b)$. Prove that f is uniformly continuous on (a, b) using the Mean Value Theorem.

   Solution:

   Let x and y be any two points in (a, b); then by the Mean Value Theorem, we know there is some point z between x and y such that

   $$\frac{f(x) - f(y)}{x - y} = f'(z).$$

   Hence since $|f'(z)| \leq M$, we know

   $$\frac{|f(x) - f(y)|}{|x - y|} \leq M,$$

   so that

   $$|f(x) - f(y)| \leq M|x - y|$$

   for any two points x and y in M.

   So let $\varepsilon > 0$ be any number, and choose $\delta = \frac{\varepsilon}{M}$. Then $|x - y| < \delta$ implies

   $$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

   So f is uniformly continuous.