Math 4310 Homework #6 Solutions

1. Prove from the definition that the following functions are continuous at every x_o in their respective domains.

   (a) $f(x) = x^3$ on $D = \mathbb{R}$.
   Solution: We have

   $$\begin{aligned} |f(x) - f(x_o)| &= |x^3 - x_o^3| \\ &= |x - x_o||x^2 + xx_o + x_o^2|. \end{aligned}$$

   So supposing that $|x - x_o| < 1$, we have

   $$|f(x) - f(x_o)| \leq |x - x_o|\big(3x_o^2 + 3|x_o| + 1\big).$$

   Now let $\varepsilon > 0$ be any number. Choose δ to be the smaller of 1 and $\frac{\varepsilon}{3x_o^2+3|x_o|+1}$. Then $|x - x_o| < \delta$ implies $|f(x) - f(x_o)| < \varepsilon$.

   (b) $f(x) = \sqrt{x}$ on $D = [0, \infty)$.
   Solution: We have

   $$\begin{aligned} |f(x) - f(x_o)| &= |\sqrt{x} - \sqrt{x_o}| \\ &= \frac{|x - x_o|}{|\sqrt{x} + \sqrt{x_o}|}. \end{aligned}$$

   If $x_o > 0$, then suppose $|x - x_o| < \frac{3x_o}{4}$; then

   $$x = |x| = |x_o + x - x_o| \geq |x_o| - |x - x_o| > \frac{x_o}{4}.$$

   Hence

   $$\sqrt{x} + \sqrt{x_o} > \sqrt{x_o/4} + \sqrt{x_o} = \frac{3}{2}\sqrt{x_o}.$$

   Therefore we have

   $$|f(x) - f(x_o) < \frac{|x - x_o|}{\frac{3}{2}\sqrt{x_o}}.$$

   Now suppose $x_o > 0$, and let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\delta \leq 3x_o/4$ and also $\delta \leq \frac{3}{2}\sqrt{x_o}\,\varepsilon$. Then whenever $|x - x_o| < \delta$, we will have

   $$|f(x) - f(x_o)| < \varepsilon.$$

   This proves continuity at $x_o > 0$.
   If $x_o = 0$, then $|f(x) - f(x_o)| = \sqrt{x}$. Let $\varepsilon > 0$ be any number, and choose $\delta = \varepsilon^2$. Then $|x - x_o| < \delta$ implies $0 \leq x < \varepsilon^2$, so that $|f(x) - f(x_o)| = \sqrt{x} < \varepsilon$. Hence we have continuity at $x_o = 0$ as well.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & x \text{ is irrational,} \\ 1 & x \text{ is rational.} \end{cases}$$

   (a) Prove that f is not continuous at any irrational number x_o.

   Solution:

   Suppose x_o is irrational; we will prove f is not continuous at x_o. In other words, we will find $\varepsilon > 0$ such that for every $\delta > 0$, there is an x with $|x - x_o| < \delta$ and $|f(x) - f(x_o)| \geq \varepsilon$.

   Choose $\varepsilon = 1$. Then for any $\delta > 0$, by Theorem 2.2.5 we can find a rational number x such that $|x - x_o| < \delta$. Hence for this x, we have $f(x) = 1$ and $f(x_o) = 0$, so that $|f(x) - f(x_o)| = 1$.

   (b) Prove that f is not continuous at any rational number x_o.

   Solution:

   We use the same technique as before; we just need to know that for any rational x_o and any $\delta > 0$, there is an irrational number x with $|x - x_o| < \delta$. Let r be any irrational number ($r = \sqrt{2}$ will work). Then for any integer n, we know that r/n is also irrational. (If r/n were rational, then $n(r/n) = r$ would also be rational, a contradiction.) Now by the Archimedean property, we know that there is some integer n such that $|r|/n < \delta$. Since x_o is rational and r/n is irrational, we know $x = x_o + r/n$ is irrational. Furthermore $|x - x_o| = |r/n| < \delta$, as desired.

   So let x_o be rational, and choose $\varepsilon = 1$. Let $\delta > 0$ be any number. By the above argument, we can find an irrational x with $|x - x_o| < \delta$. Since x_o is rational, $f(x_o) = 1$; since x is irrational, $f(x) = 0$. Hence $|f(x) - f(x_o)| = 1$.

3. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0, \end{cases}$$

   is not continuous at $x_o = 0$ in three ways:

   (a) by showing the ε-δ criterion is false;

   Solution:

   We want to show that there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there is an $x \in \mathbb{R}$ with $|x| < \delta$ and $|f(x) - 1| \geq \varepsilon$.

   Let $\varepsilon = 1$, and let $\delta > 0$ be any number. Choose $x = -\frac{\delta}{2}$. Then since $x < 0$, we know $f(x) = 0$. However we have $|x - x_o| = \delta/2 < \delta$, and we also have $|f(x) - f(x_o)| = |0 - 1| = \varepsilon$. Since this works for any $\delta > 0$, we have shown f is not continuous at $x_o = 0$.

   (b) by showing the sequence criterion is false;

   Solution:

We want to find a sequence x_k with $\lim_{k\to\infty} x_k = 0$ but $\lim_{k\to\infty} f(x_k) \neq 1$. So choose $x_k = -\frac{1}{k}$. We obviously have $\lim_{k\to\infty} x_k = 0$. Also, for every k, we know $x_k < 0$ so that $f(x_k) = 0$; hence $\lim_{k\to\infty} f(x_k) = 0$ as well.

(c) by showing the open set criterion is false.

Solution:

We want to show that there is an open set V containing $f(x_o) = 1$, such that $f^{-1}(V)$ does not contain any open interval around $x_o = 0$.

So let $V = (\frac{1}{2}, \frac{3}{2})$. Certainly V is open, since it is an open interval, and it contains $f(x_o) = 1$. However

$$f^{-1}(V) = \{x \in \mathbb{R} \mid \tfrac{1}{2} < f(x) < \tfrac{3}{2}\} = [0, \infty),$$

which does not contain any open interval around $x_o = 0$. Hence f violates the open set criterion at x_o.

4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is any function.

(a) Prove that if A and B are any subsets of $\mathbb{R}$, then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. (Hint: show that the left side is a subset of the right side, and that the right side is a subset of the left side.)

Solution:

Suppose $x \in f^{-1}(A \cup B)$. Then $f(x) \in A \cup B$, so either $f(x) \in A$ or $f(x) \in B$. Thus either $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. Thus $x \in f^{-1}(A) \cup f^{-1}(B)$. Therefore $f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B)$.

Now suppose $x \in f^{-1}(A) \cup f^{-1}(B)$. Then either $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. In the first case, $f(x) \in A$; in the second case, $f(x) \in B$. Hence $f(x) \in A \cup B$. So $x \in f^{-1}(A \cup B)$. Therefore $f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B)$.

(b) Prove that if A and B are any subsets of $\mathbb{R}$, then $f(A \cup B) \subset f(A) \cup f(B)$.

Solution: Suppose $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. So either $x \in A$ or $x \in B$. In the first case, $y = f(x) \in f(A)$; in the second case, $y = f(x) \in f(B)$. Either way, $y \in f(A) \cup f(B)$. So $f(A \cup B) \subset f(A) \cup f(B)$.

(c) Give an explicit example of sets A and B and a function f such that $f(A \cup B) \neq f(A) \cup f(B)$.

Solution: There actually isn't one; in fact $f(A \cup B) = f(A) \cup f(B)$ for all sets A and B, and all functions f. To prove this, we just need to show $f(A) \cup f(B) \subset f(A \cup B)$. So suppose $y \in f(A) \cup f(B)$. Then either $y \in f(A)$, or $y \in f(B)$. In the first case, $y = f(x)$ for some $x \in A$; in the second case, $y = f(x)$ for some $x \in B$. Either way, $y = f(x)$ for some $x \in A \cup B$. Hence $y \in f(A \cup B)$. This proves $f(A) \cup f(B) \subset f(A \cup B)$.

5. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is a function which is continuous at every point, then for any $r \in \mathbb{R}$, the set $f^{-1}\{r\}$ is closed.

Solution:

There are a few ways to do this; one is to prove in general that inverse images of closed sets are closed, using the fact that inverse images of open sets are open. A more direct way is to just prove that $f^{-1}\{r\}$ contains all its limit points, which is what we'll do here.

So suppose x is a limit point of $f^{-1}\{r\}$; then there is a sequence x_k of points with $x_k \in f^{-1}\{r\}$ with $\lim_{k\to\infty} x_k = x$. By definition of $f^{-1}\{r\}$, we know $f(x_k) = r$ for all k. Hence since f is continuous at every point, we know $f(x) = \lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} r = r$. Thus again by definition of $f^{-1}\{r\}$, we know $x \in f^{-1}\{r\}$.

6. (a) Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a sequence (x_k) such that $\lim\limits_{k\to\infty} f(x_k)$ exists, but $\lim\limits_{k\to\infty} x_k$ does not exist.

   Solution:

   Let $f(x) = x^2$. Let $x_k = (-1)^k$. Then $f(x_k) = 1$ for all k, and hence $\lim_{k\to\infty} f(x_k) = 1$. But $\lim_{k\to\infty} x_k$ does not exist.

   (b) Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and an open set $U \subset \mathbb{R}$ such that $f(U)$ is not open.

   Solution:

   Again let $f(x) = x^2$. Let U be the open interval $(-1, 1)$. Then $f(U) = [0, 1)$, which is not open.

7. Say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *schmontinuous* at x_o if for every $\delta > 0$, there exists an $\varepsilon > 0$ such that whenever $|x - x_o| < \delta$, we have $|f(x) - f(x_o)| < \varepsilon$.

   (a) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is bounded, (i.e., there exists M such that for every $x \in \mathbb{R}$, we have $|f(x)| \le M$), then f is schmontinuous at every $x_o \in \mathbb{R}$.

   Solution:

   Suppose there is such an M. For any $\delta > 0$, let $\varepsilon > 2M$. Then whenever $|x - x_o| < \delta$ (in fact any time at all) we have by the triangle inequality that

   $$|f(x) - f(x_o)| \le |f(x)| + |f(x_o)| \le M + M = 2M < \varepsilon.$$

   Since this is true for any $\delta > 0$, we have proven schmontinuity.

   (b) Give an explicit description of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is not schmontinuous at $x_o = 0$. Prove this using the negation of schmontinuity.

   Solution:

   The negation of 'schmontinuous' is, there exists a $\delta > 0$ such that for every $\varepsilon > 0$, there is a point x with $|x - x_o| < \delta$ but $|f(x) - f(x_o)| \ge \varepsilon$. Obviously the only way for a function to be not schmontinuous is to be unbounded. So define

   $$f(x) = \begin{cases} \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   Choose $\delta = 1$, and let $\varepsilon > 0$ be any number. Choose a positive x to be smaller than $\frac{1}{\varepsilon}$ and also smaller than 1. Then $|f(x) - f(0)| = 1/x > \varepsilon$ and $|x - 0| < 1 = \delta$, as desired.

8. Prove that if f is continuous on a compact set K, then either $f(x) = 0$ for some $x \in K$, or there is a number $\varepsilon > 0$ such that $|f(x)| \geq \varepsilon$ for all $x \in K$.

   Solution: Suppose there is no number $\varepsilon > 0$ such that $|f(x)| \geq \varepsilon$ for all $x \in K$. We want to prove that $f(x) = 0$ for some $x \in K$. We know that for every $\varepsilon > 0$, there is some $x \in K$ with $|f(x)| < \varepsilon$. So for each $k \in \mathbb{N}$, choose $x_k \in K$ such that $|f(x_k)| < \frac{1}{k}$.

   Now since K is compact, there is a subsequence (x_{n_k}) which converges to some $x \in K$. Since f is continuous at x, we know $f(x) = \lim_{k\to\infty} f(x_{n_k}) = 0$.

   Thus either there is some $\varepsilon > 0$ with $|f(x)| \geq \varepsilon$ for all $x \in K$, or there is some $x \in K$ with $f(x) = 0$.

9. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x_o \in \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous at $f(x_o)$. Prove that $g \circ f$, defined by $(g \circ f)(x) = g\big(f(x)\big)$, is continuous at x_o.

   Solution: Let $\varepsilon > 0$. Since g is continuous at $f(x_o)$, there is some $\eta > 0$ such that, for all u with $|u - f(x_o)| < \eta$, we have $|g(u) - g\big(f(x_o)\big)| < \varepsilon$.

   Now since f is continuous at x_o, there is some $\delta > 0$ such that for all x with $|x-x_o| < \delta$, we have $|f(x) - f(x_o)| < \eta$. Hence

   $$|x - x_o| < \delta \Longrightarrow |f(x) - f(x_o)| < \eta \Longrightarrow |g\big(f(x)\big) - g\big(f(x_o)\big)| < \varepsilon.$$

   Since we can find such a δ for any ε, we know $g \circ f$ is continuous at x_o.

10. Prove directly from the definition of uniform continuity that $f(x) = \sqrt{x}$ is not uniformly continuous on $[0, \infty)$.

    Solution: This one is harder to prove than I expected. (Actually, it's false, which makes it impossible to prove.)