Math 4310 Homework #5 Solutions

1. Give an example of closed sets $C_1, C_2, C_3, \cdots$ such that the countable union

$$C = \bigcup_{j=1}^{\infty} C_j$$

is not closed. (Prove your answer, i.e., find explicitly a limit point of your C which is not in C.)

Solution:

Let $C_j = [\frac{1}{j}, \infty)$ for each $j \in \mathbb{N}$. Each C_j is a closed set since its complement $\mathbb{R} \backslash C_j = (-\infty, \frac{1}{j})$ is an open interval.

First let us prove that the union is $C = \bigcup_{j=1}^{\infty} C_j = (0, \infty)$: if $x > 0$ then there is an integer j such that $\frac{1}{j} < x$, which implies $x \in U_j$ for this j; hence $x \in U$. On the other hand, if $x \le 0$ then $x \notin C_j$ for every j and hence $x \notin C$.

Now we prove C is not closed. We claim $x = 0$ is a limit point of C, and to prove this, we let $\varepsilon > 0$ be any number and choose $y = \frac{\varepsilon}{2}$. Then $y \in C$ and $|y - x| = |\frac{\varepsilon}{2} - 0| = \frac{\varepsilon}{2} < \varepsilon$. Hence for every $\varepsilon > 0$, there is a $y \in C$ with $y \neq 0$ and $|y - x| < \varepsilon$, which is the definition of limit point. So C does not contain one of its limit points, and hence C is not closed.

2. Prove that every open set in $\mathbb{R}$ is the union of a countable number of disjoint intervals, using the following technique. Use only the definition of open set and the order properties of real numbers (e.g., every set with an upper bound has a least upper bound).

Suppose U is an open set in $\mathbb{R}$. For every $x \in U$, let $b_x = \sup\{y \,|\, [x, y) \subset U\}$ and let $a_x = \inf\{y \,|\, (y, x] \subset U\}$. We may have $a_x = -\infty$ and $b_x = +\infty$, depending on the set.

(a) Prove that $x \in (a_x, b_x)$ and $(a_x, b_x) \subset U$.

Solution: To simplify notation a bit, for any $x \in U$, let A_x be the set $\{y \,|\, (y, x] \subset U\}$, so that $a_x = \inf A_x$. Since U is an open set with $x \in U$, there is some open interval (a, b) with $x \in (a, b)$ and $(a, b) \subset U$. Hence also $(a, x] \subset (a, b) \subset U$, which implies $a \in A_x$ by definition of A_x. Thus $a_x \le a$, since a_x is a lower bound of A_x. Since $x \in (a, b)$, we know $a < x$. Thus $a_x \le a$ and $a < x$ imply that $a_x < x$.

The same argument, with all inequalities reversed, proves $b_x > x$. Thus $x \in (a_x, b_x)$.

Now suppose $z \in (a_x, b_x)$. We want to prove $z \in U$. We know either $z < x$ or $z = x$ or $z > x$. If $z = x$ then $z \in U$ since we just proved $x \in U$. If $z < x$, then since $z > a_x$, we know z is not a lower bound for $\{y \,|\, (y, x] \subset U\}$; this means there is some y in this set with $y < z$, i.e., that there is a $y < z$ with $(y, x] \subset U$. Since $z > y$ and $z < x$, we have $z \in (y, x]$ and thus $z \in U$. Similarly if $z > x$ then $z \in U$ by the same reasoning.

(b) Prove that $a_x \notin U$ and $b_x \notin U$.

Solution: If a_x were in U, then there would be some open interval (p, q) with $a_x \in (p, q)$ and $(p, q) \subset U$, by definition of open set. Now since $q > a_x$, we know q is not a lower bound of the set A_x, since a_x is the greatest lower bound. This means there is some $y < q$ with $y \in A_x$, i.e., with $(y, x] \subset U$. Since $y \in A_x$ and a_x is a lower bound of A_x, we know $a_x \leq y$. Thus $y > p$, and thus we know $(p, q) \cup (y, x] = (p, x]$. Since $(p, q) \subset U$ and $(y, x]$ subset U, we know $(p, x] \subset U$. Hence $p \in A_x$ by definition of A_x. But this is impossible since $p < a_x$ and a_x is the lower bound of A_x. Hence a_x is not in U.

Similarly b_x is not in U.

(c) Prove that if $y \in (a_x, b_x)$, then $a_y = a_x$ and $b_y = b_x$. Conclude that for any $x, y \in U$, the intervals (a_x, b_x) and (a_y, b_y) are either the same or disjoint.

Solution: Let x and y be any elements of U. Suppose $y \in (a_x, b_x)$. Since $(a_x, b_x) \subset U$, we know $(a_x, y] \subset (a_x, b_x) \subset U$, so that $a_x \in A_y$. This implies $a_y \leq a_x$. Similarly $a_y \in A_x$, so that $a_x \leq a_y$. The only way we have both $a_x \leq a_y$ and $a_y \leq a_x$ is if $a_x = a_y$. Similarly $b_x = b_y$.

As a result, if (a_x, b_x) and (a_y, b_y) have any point z in common, then $z \in (a_x, b_x)$ implies $(a_x, b_x) = (a_z, b_z)$, while $z \in (a_y, b_y)$ implies $(a_y, b_y) = (a_z, b_z)$. Hence $(a_x, b_x) = (a_y, b_y)$. So either the sets either have no elements in common, or they must be the same.

(d) By choosing a rational number in each interval (a_x, b_x), show that there at most countably many distinct intervals.

Solution: If U is nonempty, then it must contain at least one interval, and every interval contains at least one rational number (in fact infinitely many rational numbers).

Hence the set $\mathbb{Q} \cap U$, the set of all rational numbers contained in U, is nonempty. It is countable, since it is a subset of the rationals which are countable and is also infinite. Now for every rational number $q \in U$, let $I_q = (a_q, b_q)$. So we have a function from the rationals to the open intervals of U, given by $F(q) = I_q$. This function F is not one-to-one (since for every rational $p \in I_q$, we have $F(p) = I_p = I_q$), but it is onto (since every interval I_x contains a rational number q with $F(q) = I_q = I_x$). Thus we have a function from the natural numbers onto the intervals that make up U, which means the image of this function is either finite or countable. Hence there are either finitely many or countably many intervals making up U.

3. Prove that the only nonempty subset of $\mathbb{R}$ which is both open and closed is $\mathbb{R}$ itself.

Solution: Using the structure theorem from Problem 2, this is easy. Suppose U is a set which is both open and closed and nonempty. Take any $x \in U$ (since U is nonempty); then $I_x \subset U$ since U is open.

Assume (to get a contradiction) that $a_x \neq -\infty$. We will prove that a_x is a limit point of U. For any $\varepsilon > 0$, we know $a_x + \varepsilon$ is not a lower bound of A_x, which means there is a $y \in A_x$ with $a_x \leq y < a_x + \varepsilon$. The fact that $y \in A_x$ means by definition that

$(y, x] \subset U$, so that any number z between y and $a_x + \varepsilon$ is in U. Since for any ε we can find $z \in U$ with $a_x \le z < a_x + \varepsilon$, we see that a_x is a limit point of U. But we know $a_x \notin U$, which contradicts the fact that U is closed. Hence we must have $a_x = -\infty$. Similarly we must have $b_x = +\infty$. So $I_x = \mathbb{R}$, and thus $U = \mathbb{R}$.

4. Prove that if E is any set (open, closed, or neither), the set of limit points of E is closed.

   Solution: Let E' denote the set of limit points of E. We want to prove that E' is closed, so let x be any limit point of E'; we want to prove $x \in E'$, i.e., that x is also a limit point of E.

   So let $\varepsilon > 0$; we want to find $z \in E$ with $z \ne x$ and $|z - x| < \varepsilon$. First, since x is a limit point of E', we can find $y \in E'$ with $y \ne x$ such that $|y - x| < \frac{\varepsilon}{2}$. Since $x \ne y$, we know $|y - x|$ is some positive number; call it δ. Since $y \in E'$, we know y is a limit point of E, so that there is some $z \in E$ with $z \ne y$ and $|z - y| < \delta$. Since $|z - y| < \delta = |x - y|$, we know $z \ne x$. Also we have

$$|x - z| = |x - y + y - z| \le |x - y| + |y - z| < \frac{\varepsilon}{2} + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   Since $\varepsilon > 0$ was arbitrary, we have proved x is a limit point of E, which means that $x \in E'$. So we have proved that any limit point of E' is also contained in E', which proves that E' is closed.

5. The set $K = [0, 1]$ is closed and bounded, hence compact. Show that the family of open sets $U_x = (x - 1/2, x + 1/2)$, for $x \in \mathbb{R}$, covers K. Find a subcover with the fewest number of elements possible.

   Solution: We have $\bigcup_{x\in\mathbb{R}} U_x = \mathbb{R}$, so that in particular $K \subset \bigcup_{x\in\mathbb{R}}$. Thus the family U_x covers K.

   We cannot cover K with one set U_x, since if we could, we would have $0 > x - \frac{1}{2}$ and $1 < x + \frac{1}{2}$. The former implies $x < \frac{1}{2}$ while the latter implies $x > \frac{1}{2}$, and this is impossible. However, we can cover K with two sets, for example $U_{1/3}$ and $U_{2/3}$, since

$$U_{1/3} \cup U_{2/3} = (-\tfrac{1}{6}, \tfrac{5}{6}) \cup (\tfrac{1}{6}, \tfrac{7}{6}) = (-\tfrac{1}{6}, \tfrac{7}{6})$$

   contains $[0, 1] = K$.

6. Prove that an arbitrary intersection of compact sets is compact. Prove that a finite union of compact sets is compact.

   Solution: Suppose J is any set and for each $j \in J$ we have a compact set K_j. Let $K = \bigcap_{j\in J} K_j$. We know that a set is compact if and only if it is closed and bounded. So each K_j is closed, and therefore K is also closed. Also each K_j is bounded, so any particular K_j is contained in some interval $[-M, M]$; hence K is also contained in $[-M, M]$ and is therefore bounded. As a result K must be compact.

   The same technique works for a finite union $K = K_1 \cup \cdots \cup K_n$ of compact sets. First every K_j is closed, and a finite union of closed sets is closed; thus K is closed.

Also each K_j is contained in some interval $[-M_j, M_j]$, and therefore setting $M = \max\{M_1, \cdots, M_n\}$ we know that every K_j is contained in $[-M, M]$. Thus also $K \subset [-M, M]$, so K is bounded. Since K is both closed and bounded, it is compact.

7. Prove that the set of real numbers in $[0, 1]$ whose decimal expansions contain only the digits 3 and 5 is compact. (Hint: write this set as an intersection of closed sets.)

   Solution: Since the desired set is a subset of $[0, 1]$, it is bounded. Thus to prove it's compact, we just have to prove it's closed.

   Let $E_1 = [0.3, 0.4] \cup [0.5, 0.6]$; then every number in E_1 except for 0.4 and 0.6 has either a 3 or 5 as its first decimal place, and furthermore E_1 is closed. Let

$$E_2 = [0.33, 0.34] \cup [0.35, 0.36] \cup [0.53, 0.54] \cup [0.55, 0.56].$$

   Then E_2 is also closed, and the first two digits of every number in E_2 are either 3 or 5, except for the right endpoints of E_2. Observe that the right endpoints of E_1 are not in E_2.

   Continue in this way, obtaining for each $k \in \mathbb{N}$ a set E_k consisting of the union of 2^k closed intervals (hence closed, since a union of finitely many closed sets is closed), such that every element of E_k has a decimal expansion whose first k digits are either 3s or 5s, except for the right endpoints of E_k. Observe that the right endpoints of every E_k are not contained in E_{k+1}.

   Now set $E = \bigcap_{k=1}^{\infty} E_k$. We know E is closed, since arbitrary intersections of closed sets are closed. Now we want to prove E is the desired set.

   If $x \in E$, then $x \in E_k$ for every k. Thus x cannot be a right endpoint of any E_k (since if it were, x would not be in E_{k+1}). Thus for every k, the first k digits of the decimal expansion of x are 3s and 5s. Since this is true for every k, the full decimal expansion of x must consist only of 3s and 5s.

   Now let us prove that every x whose decimal expansion consists only of 3s and 5s is in E. We know such an x will be in E_1 since the first digit must be either 3 or 5; it will be in E_2 since the second digit is also either 3 or 5; similarly it will be in every E_k. Thus $x \in E$.