Math 4310 Homework #4 Solutions

1. A sequence of real numbers $x_1, x_2, x_3, \cdots$ has limit x if, for every $\varepsilon > 0$, there is an integer N such that for every $k \geq N$, we have $|x_k - x| < \varepsilon$.

   Prove using only the definition of limit that if $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$, then

   $$\lim_{k\to\infty}(x_k + y_k) = x + y$$

   and

   $$\lim_{k\to\infty} x_k y_k = xy.$$

   (You do not need to use Cauchy sequences explicitly.)

   Solution: The techniques here are precisely the same as for proving that the sums and products of Cauchy sequences are Cauchy.

   First let us show that $\lim_{k\to\infty}(x_k + y_k) = x + y$. Let $\varepsilon > 0$ be any number. Since $\lim_{k\to\infty} x_k = x$, there is an $N_1 \in \mathbb{N}$ such that whenever $k \geq N_1$, we have $|x_k - x| < \frac{\varepsilon}{2}$. Similarly we can find $N_2 \in \mathbb{N}$ such that whenever $k \geq N_2$, we have $|y_k - y| < \frac{\varepsilon}{2}$. Thus if we set $N = \max\{N_1, N_2\}$, then for any $k \geq N$, we have

   $$|(x_k + y_k) - (x + y)| = |(x_k - x) + (y_k - y)| \leq |x_k - x| + |y_k - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

   Now we show that $\lim_{k\to\infty}(x_k y_k) = xy$. Let $\varepsilon > 0$. Since $\lim_{k\to\infty} x_k = x$, we can choose N_1 so that whenever $k \geq N_1$, we have

   $$|x_k - x| < \frac{\varepsilon}{2(|y| + 1)}$$

   Similarly we can choose N_2 so that whenever $k \geq N_2$, we have

   $$|y_k - y| < \frac{\varepsilon}{2(|x| + 1)}.$$

   Finally we can choose N_3 so that $k \geq N_3$ implies $|x_k| \leq |x| + 1$. Then setting $N = \max\{N_1, N_2, N_3\}$, we see that whenever $k \geq N$, we have

   $$\begin{aligned} |x_k y_k - xy| &= |x_k y_k - x_k y + x_k y - xy| \\ &\leq |x_k|\,|y_k - y| + |y|\,|x_k - x| \\ &\leq \big(|x| + 1\big)\,|y_k - y| + \big(|y| + 1\big)\,|x_k - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

2. You've proved in the last homework that every decimal expansion

   $$(0, 0.a_1, 0.a_1a_2, 0.a_1a_2a_3, \cdots)$$

   is a Cauchy sequence.

(a) Prove that every real number x in $[0,1)$, the interval containing 0 but not 1, has a decimal expansion. (To find the first decimal place, show that every real number must lie in one of the intervals $[0,0.1)$, $[0.1,0.2)$, $[0.2,0.3)$, etc. Continue in this way to find subsequent terms. Prove that the resulting sequence converges to x.)

Solution: Take any real x with $x \geq 0$ and $x < 1$. Then by the order property, we know exactly one is true: either $x < 0.1$, or $x = 0.1$, or $x > 0.1$. If $x < 0.1$, set the first decimal place to be $a_1 = 0$. If $x \geq 0.1$, then we check whether $x < 0.2$ or $x \geq 0.2$. If $0.1 \leq x < 0.2$, then set the first decimal place as $a_1 = 1$; if $x \geq 0.2$, then we check 0.3. Continuing in this way, we will eventually stop at the 0.9 comparison, so that we will determine a_1 to be one of $\{0, 1, 2, \cdots, 9\}$. Also observe that we always have $\frac{a_1}{10} \leq x < \frac{a_1+1}{10}$, which implies that $|x - \frac{a_1}{10}| < \frac{1}{10}$. Now we divide up the interval $[\frac{a_1}{10}\frac{a_1+1}{10})$ into ten pieces in the same way. First compare x with $\frac{a_1}{10} + \frac{1}{100}$; if it's smaller, set $a_2 = 0$. If it's larger or equal, then compare with $\frac{a_1}{10} + \frac{2}{100}$. Continuing in this way, we obtain $a_2 \in \{0, 1, 2, \cdots, 9\}$. Furthermore we know that $|x - \left(\frac{a_1}{10} + \frac{a_2}{10^2}\right)| < \frac{1}{10^2}$.

Continuing in this way, we obtain the decimal expansions

$$\begin{aligned} q_1 &= \frac{a_1}{10} \\ q_2 &= \frac{a_1}{10} + \frac{a_2}{10^2} \\ q_3 &= \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3}, \end{aligned}$$

etc. Each q_k is a rational number satisfying $|x - q_k| < \frac{1}{10^k}$. Hence we can prove that $\lim_{k\to\infty} q_k = x$, since for any $\varepsilon > 0$, choosing $N > \log_{10}(1/\varepsilon)$ implies that whenever $k \geq N$, we have $|x - q_k| < \frac{1}{10^N} < \varepsilon$.

(b) Show that each real number has only one decimal expansion, except for those whose decimal expansions terminate in a string of zeroes (which have two different decimal expansion).

Solution: Suppose we have two equivalent decimal expansions $\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=1}^{\infty} \frac{b_k}{10^k}$; we want to determine whether $a_k = b_k$ for all k. So consider for each $n \in \mathbb{N}$ the numbers

$$r_n = \sum_{k=1}^{n} \frac{a_k}{10^k} - \sum_{k=1}^{n} \frac{b_k}{10^k} = \sum_{k=1}^{n} \frac{a_k - b_k}{10^k}.$$

Suppose that the expansions $\{a_k\}$ and $\{b_k\}$ agree up to the j^{th} place and not at the $(j+1)^{\text{st}}$ place. For $n > j+1$, we have (by the reversed triangle inequality) that

$$
\begin{aligned}
|r_n| &= \left|\sum_{k=1}^{n} \frac{a_k - b_k}{10^k}\right| \\
&= \left|\sum_{k=j+1}^{n} \frac{a_k - b_k}{10^k}\right| \\
&\geq \frac{|a_{j+1} - b_{j+1}|}{10^{j+1}} - \left|\sum_{k=j+2}^{n} \frac{a_k - b_k}{10^k}\right| \\
&\geq \frac{|a_{j+1} - b_{j+1}|}{10^{j+1}} - \sum_{k=j+2}^{n} \frac{|a_k - b_k|}{10^k} \\
&\geq \frac{|a_{j+1} - b_{j+1}|}{10^{j+1}} - \sum_{k=j+2}^{n} \frac{9}{10^k} \\
&= \frac{|a_{j+1} - b_{j+1}|}{10^{j+1}} - \left(\frac{1}{10^{j+1}} - \frac{1}{10^n}\right) \\
&= \frac{|a_{j+1} - b_{j+1}| - 1}{10^{j+1}} + \frac{1}{10^n}.
\end{aligned}
$$

Therefore we know that if $|a_{j+1} - b_{j+1}| > 1$, then $r_n \geq \frac{1}{10^{j+1}}$ for all n, and hence we don't have $\lim_{n\to\infty} r_n = 0$, which means the two sequences are not equivalent (a contradiction to our original assumption). If however $|a_{j+1} - b_{j+1}| = 1$, then we may have $|r_n| = \frac{1}{10^n}$ if all the inequalities are actually equalities. Our next step is to determine when this happens, and to do this we want to remove the absolute values.

So we must have $a_{j+1} = b_{j+1} \pm 1$. Suppose without loss of generality that $a_{j+1} = b_{j+1} + 1$. Then

$$r_n = \sum_{k=1}^{n} \frac{a_k - b_k}{10^k} = \frac{1}{10^{j+1}} + \sum_{k=j+2}^{n} \frac{a_k - b_k}{10^k}.$$

We thus have

$$
\begin{aligned}
r_n &= \frac{1}{10^{j+1}} - \sum_{k=j+2}^{n} \frac{9}{10^k} + \sum_{k=j+2}^{n} \frac{a_k - b_k + 9}{10^k} \\
&= \frac{1}{10^n} + \sum_{k=j+2}^{n} \frac{a_k - b_k + 9}{10^k}.
\end{aligned}
$$

We know that $a_k - b_k + 9 \geq 0$ for all k. If $a_m - b_m + 9 > 0$ for *any* $m \geq j + 2$, then we will have

$$r_n \geq \frac{1}{10^n} + \frac{a_m - b_m + 9}{10^m}$$

for *every* $n \geq m$, and hence again we cannot have $\lim_{n\to\infty} r_n = 0$. So we must have $a_m - b_m + 9 = 0$ for every $m \geq j + 2$.

Since a_m and b_m are both in the sets $\{0, 1, \cdots, 9\}$, the only way this happens is if $a_m = 0$ and $b_m = 9$ for all $m \geq j+2$. Thus there are only two different decimal expansions that can be equivalent:

$$0.a_1a_2a_3 \cdots a_j a_{j+1} 99999 \cdots = 0.a_1a_2a_3 \cdots a_j(a_{j+1}+1)00000 \cdots$$

(c) Show that the set of real numbers with more than one decimal expansion is countable.

Solution: We could count the numbers with terminating decimal expansions explicitly: there are nine numbers which terminate at the first decimal place, ninety numbers which terminate at the second decimal place, and in general $9 \times 10^{k-1}$ numbers terminating at the k^{th} decimal place. So clearly we can count based on k.

Alternatively we could just notice that every terminating decimal is a particular rational number, so that the set of all reals with terminating decimals is a subset of the rationals (in particular, those which can be written with a power of 10 in the denominator). Since the rationals are countable, so is this subset.

3. Textbook exercise, 3.1.3: 1

Compute the sup, inf, limsup, liminf, and all the limit points of the following sequences $x_1, x_2, \ldots$ where

(a) $x_n = 1/n + (-1)^n$

Solution: All the even terms are positive while all the odd terms are negative; hence the supremum of all terms is the supremum of the even terms, while the infimum of all terms is the infimum of the odd terms. So $\sup\{x_n\} = \sup\{1/(2n) + 1\} = \frac{3}{2}$ while $\inf\{x_n\} = \inf\{1/(2n-1) - 1\} = -1$.

The subsequence of even terms is $x_{2k} = 1/(2k) + 1$ which converges to 1, while the subsequence of odd terms is $x_{2k-1} = 1/(2k-1) - 1$ which converges to -1. Hence 1 and -1 are limit points. There are no other limit points, since for any $\varepsilon > 0$, if $N > 1/\varepsilon$, then every term of the sequence beyond N is within ε of either 1 and -1.

So $\limsup_{n\to\infty} x_n = \sup\{1, -1\} = 1$ while $\liminf_{n\to\infty} x_n = \inf\{1, -1\} = -1$.

(b) $x_n = 1 + (-1)^n/n$.

Solution: The even terms are all larger than 1 while the odd terms are all smaller than 1. Hence $\sup\{x_n\} = \sup\{1+1/(2n)\} = \frac{3}{2}$ and $\inf\{x_n\} = \inf\{1-1/(2n-1)\} = 0$.

The sequence converges to 1 since $|x_n - 1| = |(-1)^n/n| = 1/n$, and for any $\varepsilon > 0$, choosing $N > 1/\varepsilon$ implies that for $n \geq N$ we have $|x_n - 1| < 1/N < \varepsilon$.

Since the sequence converges, the only limit point is $\{1\}$, so this is the limsup and the liminf as well.

(c) $x_n = (-1)^n + 1/n + 2\sin n\pi/2$.

Solution: Every positive integer n can be expressed as exactly one of the following: $n = 4k$, $n = 4k - 1$, $n = 4k - 2$, $n = 4k - 3$, for some positive integer k. So we have

$$x_{4k} = 1 + 1/(4k), \quad x_{4k-1} = -3 + 1/(4k-1),$$
$$x_{4k-2} = 1 + 1/(4k-2), \quad x_{4k-3} = 1 + 1/(4k-3).$$

All the terms except for $n = 4k-1$ are positive, so for the supremum we only need to consider those. All the $n = 4k-1$ terms are negative, so for the infimum we only consider those. Thus $\sup\{x_n\} = \sup\{1+1/(4k), 1+1/(4k-2), 1+1/(4k-3)\} = 2$, while $\inf\{x_n\} = \inf\{-3 + 1/(4k-1)\} = -3$.

We clearly have

$$\lim_{k\to\infty} x_{4k} = 1, \quad \lim_{k\to\infty} x_{4k-1} = -3, \quad \lim_{k\to\infty} x_{4k-2} = 1, \quad \lim_{k\to\infty} x_{4k-3} = 1.$$

Thus the limit points are 1 and -3, so that the limsup is 1 and the liminf is -3.

4. Textbook problem, 3.1.3: 2.

   If a bounded sequence is the sum of a monotone increasing and a monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing) does it follow that the sequence converges? What if $\{y_n\}$ and $\{z_n\}$ are bounded?

   Solution:

   In general, the answer is 'no.' Take for example $y_n = 3n + (-1)^n$ and $z_n = -3n$. Then y_n is monotone increasing, since $y_{n+1} - y_n = 3(n+1) + (-1)^{n+1} - 3n - (-1)^n = 3 - 2(-1)^n \geq 1$. Clearly z_n is monotone decreasing. But $x_n = y_n + z_n = (-1)^n$ does not converge.

   On the other hand, if $\{y_n\}$ is bounded, then it must converge to some number y (since any bounded monotone sequence converges). Similarly z_n must converge to some number z. Hence x_n converges to $y + z$.

5. Textbook problem, 3.1.3: 4.

   Prove $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$.

   Solution:

   In general, if $U \subset V$, then $\sup U \leq \sup V$ since $\sup V$ is an upper bound of V and hence also U, while $\sup U$ is less than or equal to any upper bound of U. Therefore, since $A \subset A \cup B$ we have $\sup A \leq \sup(A \cup B)$. Also since $A \cap B \subset A$, we have $\sup(A \cap B) \leq \sup A$.

6. Textbook problem, 3.1.3: 5.

   Prove $\limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$ if both lim sups are finite, and give an example where equality does not hold.

   Solution:

For each $n \in \mathbb{N}$, let $a_n = \sup\limits_{m>n} x_m$ and $b_n = \sup\limits_{m>n} y_m$. Then by definition of limsup, we have $\limsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} a_n$ and $\limsup\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} b_n$.

Furthermore for every $n \in \mathbb{N}$ and every $k > n$, we have $x_k \leq a_n$ (by definition of supremum). Similarly $y_k \leq b_n$.

Thus for every n and every $k > n$, we have $x_k + y_k \leq a_n + b_n$. Since this is true for every $k > n$, we know that $a_n + b_n$ is an upper bound for the set of $(x_k + y_k)$ with $k > n$. So

$$\sup_{k>n}(x_k + y_k) \leq a_n + b_n,$$

for every $n \in \mathbb{N}$.

Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}(x_n + y_n) &= \lim_{n\to\infty} \sup_{k>n}(x_k + y_k) \\ &\leq \lim_{n\to\infty}(a_n + b_n) \\ &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \\ &= \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,\end{aligned}$$

as desired.

Now we want an example where the two are not equal. First observe that if x_n and y_n are both convergent sequences, then so is $x_n + y_n$, so that we must have equality. Hence the only way to get a strict inequality is to have at least one nonconvergent sequence. A simple example is $x_n = (-1)^n$ and $y_n = -(-1)^n$. Then $\limsup\limits_{n\to\infty} x_n = 1$ and $\limsup\limits_{n\to\infty} y_n = 1$, but

$$\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty}(0) = 0.$$