

Math 4310 Homework #3 Solutions

1. Prove the triangle inequality

$$|x + y| \leq |x| + |y|$$

for numbers $x, y \in \mathbb{Q}$, by considering cases.

Solution: There are three absolute values, with two possibilities for each term (either nonnegative or negative) and thus eight possible cases to consider.

- (a) $x \geq 0, y \geq 0, x + y \geq 0$: then the left and right sides are both $x + y$, so we have equality.
- (b) $x \geq 0, y \geq 0, x + y < 0$: this is impossible, since the sum of nonnegative numbers is also nonnegative.
- (c) $x \geq 0, y < 0, x + y \geq 0$: then the left side is $x + y$ and the right side is $x - y$.

$$\begin{array}{ll} y < 0 & \text{by assumption} \\ 2y < 0 & \text{multiplying both sides by 2} \\ y < -y & \text{subtracting } y \text{ from both sides} \\ x + y < x - y & \text{adding } x \text{ to both sides} \\ |x + y| < |x| + |y|. & \end{array}$$

- (d) $x \geq 0, y < 0, x + y < 0$: the left side is $-(x + y)$ and the right side is $x - y$. We have

$$0 \leq x \implies 0 \leq 2x \implies -x \leq x \implies -x - y \leq x - y \implies |x + y| \leq |x| + |y|.$$

- (e) $x < 0, y \geq 0, x + y \geq 0$: this is the same as case (c) with x and y switched.
- (f) $x < 0, y \geq 0, x + y < 0$: this is the same as case (d) with x and y switched.
- (g) $x < 0, y < 0, x + y \geq 0$: this is impossible.
- (h) $x < 0, y < 0, x + y < 0$: then the left side is $-(x + y)$ and the right side is $-x - y$, so we have equality.

2. Prove directly from the definition that the sequence

$$a_n = \frac{3 + n}{2n + 1}, n \geq 1$$

is a Cauchy sequence. In other words, for any ε , determine explicitly the integer N such that whenever $j \geq N$ and $k \geq N$, we have $|a_j - a_k| < \varepsilon$.

Solution: We first compute $|a_j - a_k|$:

$$\begin{aligned} |a_j - a_k| &= \left| \frac{3+j}{2j+1} - \frac{3+k}{2k+1} \right| \\ &= \left| \frac{5(k-j)}{(2j+1)(2k+1)} \right| \\ &= \frac{5}{(2j+1)(2k+1)} |k-j| \\ &\leq \frac{5(j+k)}{(2j+1)(2k+1)} \\ &\leq \frac{5}{2(2k+1)} + \frac{5}{2(2j+1)}. \end{aligned}$$

Now if $j \geq N$ and $k \geq N$, then

$$|a_j - a_k| \leq \frac{5}{2(2N+1)} + \frac{5}{2(2N+1)} = \frac{5}{2N+1}.$$

We want this to be less than an $\varepsilon > 0$. Solving $\frac{5}{2N+1} < \varepsilon$, we obtain $N > \frac{5-\varepsilon}{2\varepsilon}$.

Now going the other way around, let ε be any positive number. Choose an integer $N > \frac{5-\varepsilon}{2\varepsilon}$. Then for every $j \geq N$ and $k \geq N$, we have $|a_j - a_k| < \varepsilon$. So (a_n) is a Cauchy sequence.

3. Find the negation of the definition of Cauchy sequence, and prove directly from the negation that

$$a_n = \cos \pi n, n \geq 1$$

is not a Cauchy sequence.

Solution: A sequence (a_n) is Cauchy if, for every $\varepsilon > 0$, there is an integer N such that for every $j \geq N$ and for every $k \geq N$, we have $|a_j - a_k| < \varepsilon$.

To negate this, we say that a sequence (a_n) is not Cauchy if there is some $\varepsilon > 0$ such that, for any integer N , there is some $j \geq N$ and some $k \geq N$ such that $|a_j - a_k| \geq \varepsilon$.

Now the particular sequence given is

$$a_n = \begin{cases} 1 & n \text{ even,} \\ -1 & n \text{ odd.} \end{cases}$$

Thus if j and k have the same parity, then $|a_j - a_k| = 0$, while if j and k have opposite parity, then $|a_j - a_k| = 2$.

So choose $\varepsilon = 2$. Then for any integer N , let $j = N$ and let $k = N + 1$. Then j and k do not have the same parity, so that $|a_j - a_k| = 2 \geq \varepsilon$. Hence (a_n) is not Cauchy.

4. Textbook 2.1.3: 5

Prove that if a Cauchy sequence x_1, x_2, \dots of rationals is modified by changing a finite number of terms, the result is an equivalent Cauchy sequence.

Solution: Let x'_1, x'_2, \dots be another Cauchy sequence, such that $x_k = x'_k$ for every $k \geq N$ (for some N). Then no matter what $\varepsilon > 0$ is, we will have $|x_k - x'_k| = 0 < \varepsilon$ as long as $k \geq N$. This proves (x_1, x_2, \dots) is equivalent to (x'_1, x'_2, \dots) .

5. Textbook 2.1.3: 6

Give a proof that any infinite decimal expansion $\pm N.a_1a_2a_3\dots$ gives a Cauchy sequence $\pm N, \pm N.a_1, \pm N.a_1a_2, \dots$

Solution: Suppose $k > j$; then

$$\begin{aligned} |x_k - x_j| &= |\pm N.a_1a_2 \cdots a_ja_{j+1} \cdots a_k - \pm N.a_1a_2 \cdots a_j| \\ &= \left| \pm N + \sum_{i=1}^k \frac{a_i}{10^i} - \pm N - \sum_{i=1}^j \frac{a_i}{10^i} \right| \\ &= \left| \sum_{i=j+1}^k \frac{a_i}{10^i} \right| \\ &\leq \sum_{i=j+1}^k \frac{9}{10^i} \\ &< \frac{1}{10^j}. \end{aligned}$$

Now for any $\varepsilon > 0$, we can find an integer N such that $\frac{1}{10^N} < \varepsilon$. (Just choose $N > \log_{10}(1/\varepsilon)$.) Then $j, k \geq N$ implies

$$|a_k - a_j| \leq \frac{1}{10^N} < \varepsilon.$$

Thus we do indeed have a Cauchy sequence.

6. Textbook 2.1.3: 7

Show that the Cauchy sequence $0.9, 0.99, 0.999, \dots$ is equivalent to $1, 1, 1, \dots$

Solution: If $a_1 = 0.9$, $a_2 = 0.99$, etc., and $b_1 = 1$, $b_2 = 1$, etc., then

$$|a_k - b_k| = \frac{1}{10^k}.$$

Now let $\varepsilon > 0$ be any number. Choose an integer $N > \log_{10}(1/\varepsilon)$; then $\frac{1}{10^N} < \varepsilon$. Thus $k \geq N$ implies

$$|a_k - b_k| = \frac{1}{10^k} \leq \frac{1}{10^N} < \varepsilon.$$

Hence the sequences (a_k) and (b_k) are equivalent.

7. Give a counterexample as requested in the Remarks for Lemma 2.2.5. Explicitly, suppose x and y are real numbers represented by Cauchy sequences $\{x_k\}$ and $\{y_k\}$. Show that the statement, “If $x \leq y$, then there is an $m \in \mathbb{N}$ such that $x_k \leq y_k$ for all $k \geq m$,” is false in general.

Solution:

Let $x = y = 0$. If $x_k = \frac{1}{k}$ and $y_k = 0$ for every $k \in \mathbb{N}$, then $x_k > y_k$ for every k . Hence the statement is false for this example.

8. Textbook 2.2.4: 3

If x is a real number, show that there exists a Cauchy sequence of rationals x_1, x_2, \dots representing x such that $x_n < x$ for all n .

Solution: For each $n \in \mathbb{N}$, Theorem 2.2.5 tells us that there is a rational number x_n with $|x - \frac{1}{n} - x_n| \leq \frac{1}{n}$. Then we have

$$\begin{aligned}
-\frac{1}{n} &\leq x - \frac{1}{n} - x_n \leq \frac{1}{n} \\
0 &\leq x - x_n \leq \frac{2}{n} \\
x_n &\leq x \leq x_n + \frac{2}{n}.
\end{aligned}$$

Thus we have $x_n \leq x$ for every n ; furthermore $\lim_{n \rightarrow \infty} x_n = x$, so that (x_1, x_2, \dots) must be a Cauchy sequence representing x .

9. Textbook 2.2.4: 7

Prove $|x - y| \geq |x| - |y|$ for any real numbers x and y .

Solution: We know

$$|x| = |y + (x - y)| \leq |y| + |x - y|.$$

Hence subtracting $|y|$ from both sides, we get

$$|x - y| \geq |x| - |y|.$$

10. Textbook 2.2.4: 9b

Prove the following inequality from the ordered field axioms:

$$2xy \leq x^2 + y^2.$$

Solution: From the previous homework, we know $a^2 \geq 0$ for any a in any field F . Hence $(x - y)^2 \geq 0$ for any x and y . Thus

$(x - y)^2 \geq 0$	by the previous homework
$(x - y)(x - y) \geq 0$	by definition of squaring
$x(x - y) - y(x - y) \geq 0$	by the distributive law
$x^2 - xy - yx + (-y)^2 \geq 0$	distributing again
$x^2 - 2xy + (-y)^2 \geq 0$	by commutativity of multiplication
$x^2 - 2xy + y^2 \geq 0$	since $(-y) = (-1) \cdot y$ and $(-1)^2 = 1$
$x^2 + y^2 \geq 2xy$	by compatibility between order and addition