Math 4310 Homework #1

1. Construct the truth table for the statements $P =$ "A or (B and C)" as well as $Q =$ "(A or B) and (A or C)," for three statements A, B, and C. (That is, write all eight possibilities for the three statements being true or false and for each combination, state whether the statement is true or false.) Show that the truth tables are the same, and conclude that P and Q are equivalent.

   Solution:

A	B	C	B and C	A or (B and C)	A or B	A or C	(A or B) and (A or C)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

   Comparing the fifth column with the eighth column, we see that P and Q always have the same values, and thus are equivalent statements.

2. Write the following English sentence explicitly in terms of quantifiers, and then write its negation.

   "Every pundit has at some point put forth an argument against your proposal."

   Solution: The original statement: "For all pundits, there exists a time and there exists an argument such that the pundit has made it against your proposal."

   Its negation: "There is at least one pundit who has never made any argument against your proposal."

3. (Corrected version.) Find the negation of the following mathematical statement.

   "For every $\varepsilon > 0$ and for every $x \in U$, there is a $y \in V$ such that $|x - y| < \varepsilon$."

   Using the negation, prove that this statement is not true for the sets $U = (0, 1)$ and $V = (-1, 1)$.

   Solution: The negation is, "There is at least one $\varepsilon > 0$ and at least one $x \in U$ such that, for every $y \in V$, we have $|x - y| \geq \varepsilon$."

   Actually, the original statement is true. For any ε and for any $x \in U$, we can set $y = x$ (since $U \subset V$). Then we have $|x - y| = 0 < \varepsilon$.

4. Textbook exercise 1.2.3, 2:

   Is the set of all finite subsets of $\mathbb{N}$ countable or uncountable? Give a proof of your assertion.

Solution: Let S_f denote the set of all finite subsets of $\mathbb{N}$, and let S_n denote the set of all nonempty subsets of $\mathbb{N}$ containing at most n elements, for any positive integer n. Then

$$S_f = \bigcup_{n=1}^{\infty} S_n,$$

a countable union of sets S_n. So if we can prove that each S_n is countable, then S_f must be countable as well (since a countable union of countable sets is countable). So we just have to prove S_n is countable.

Now S_1 consists of the subsets of $\mathbb{N}$ containing exactly one element; clearly we can list these as

$$S_1 = \Big\{ \{1\}, \{2\}, \{3\}, \cdots \Big\},$$

and this is already in one-to-one correspondence with $\mathbb{N}$. So S_1 is countable.

Next we use the induction principle, and prove that if S_n is countable, then so is S_{n+1}. Observe that every set consisting of $(n+1)$ elements is the union of some set of n elements and some set with one element. As such there is a function on sets:

$$F\colon S_n \times S_1 \to S_{n+1}$$

defined by the formula $F(A, B) = A \cup B$, and this function is onto S_{n+1}. We know that the product of two countable sets is countable, so $S_n \times S_1$ is countable. We also know that if F is a function from a countable set onto any other set, then the range of F is either finite or countable, so that S_{n+1} is either finite or countable. Finally, since S_{n+1} contains the infinite set S_1, we know S_{n+1} must be infinite, and hence countable.

We have thus proved that S_1 is countable, and that whenever S_n is countable, S_{n+1} is also countable. Hence every S_n must be countable by the principle of induction. As a result, $S_f = \cup_{n=1}^{\infty} S_n$ is countable.

5. Textbook exercise 1.2.3, 3:

   Prove that the rational numbers are countable.

   Solution: We can write, as in the hint,

$$\mathbb{Q} = \bigcup_{k=1}^{\infty} \mathbb{Q}_k,$$

   where $\mathbb{Q}_k = \{j/k : j \in \mathbb{Z}\}$. We know that $\mathbb{Q}_k$ is in one-to-one correspondence with $\mathbb{Z}$ and that $\mathbb{Z}$ is countable (from class). Hence $\mathbb{Q}_k$ is also countable, for each $k \in \mathbb{N}$. Therefore $\mathbb{Q}$ is a countable union of countable sets, and hence is also countable.

   (Alternatively, we could write the rationals as an infinite matrix and count them using the trick on page 10.)

6. Textbook exercise 1.2.3, 4.

   Show that if a countable subset is removed from an uncountable set, the remainder is still uncountable.

Solution: Let's use proof by contradiction. So assume there is an uncountable set U and a countable subset $A \subset U$, and that $U \backslash A$ is countable.

Then $U = (U \backslash A) \cup A$, the union of two countable sets; hence U must also be countable, which contradicts the assumption that U is uncountable. Hence our assumption must have been false.

7. Textbook exercise 1.2.3, 7.

   Generalize the diagonalization argument to show that 2^A has greater cardinality than A for every infinite set A.

   Solution: As with $\mathbb{N}$, we need to show that for any function $f\colon A \to 2^A$, the function f is not onto.

   So let $S = \{a \in A : a \not in f(a)\}$. Then for every $a \in A$, we either have $a \in f(a)$ or $a \notin f(a)$; in the first case, $a \notin S$, and in the second case, $a \in S$. So S and $f(a)$ do not have the same elements. This is true for every $a \in A$; hence S is not in the image of f. So f is not onto.