Math 4310 Homework #12
due Wednesday, December 12

1. Let $f(x)$ be the zero function. Prove that $f_n(x) = x^n$ does not converge uniformly to f, but that $g_n(x) = x^n(1-x)$ does converges uniformly to f. (On the interval $[0,1]$.)

   Solution:

   Obviously $|f_n(x) - f(x)| = x^n$, and since this function is increasing on $[0,1]$, the supremum occurs at $x = 1$, so that

   $$d(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1^n = 1.$$

   Since $\lim_{n\to\infty} d(f_n, f) \neq 0$, f_n does not converge uniformly to f.

   Similarly $|g_n(x) - f(x)| = x^n(1-x)$. This is a differentiable function, so it attains its maximum in $[0,1]$ either at an endpoint or at a critical point. The critical points are found from $g_n'(x) = nx^{n-1} - (n+1)x^n = 0$, so that $x = 0$ or $x = \frac{n}{n+1}$. Since $g_n(0) = 0$ and $g_n(1) = 0$, the maximum must occur at $x = \frac{n}{n+1}$, where

   $$g_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n\left(1 - \frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}.$$

   Therefore

   $$d(g_n, f) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1}.$$

   The term $(\frac{n}{n+1})^n$ converges to $1/e$, as a well-known calculus trick and L'Hopital's rule will show. But actually, all one needs to know is that $(\frac{n}{n+1})^n < 1$ for every n, since the term $\frac{1}{n+1}$ already converges to zero.

2. Prove that $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ converges uniformly to $f(x) = |x|$ on $[-1,1]$. However, $f_n'(0)$ does not converge to $f'(0)$.

   Solution:

   We compute the difference by rationalizing the denominator:

   $$|f_n(x) - f(x)| = \left|\sqrt{x^2 + \frac{1}{n}} - |x|\right| = \left|\frac{x^2 + \frac{1}{n} - x^2}{\sqrt{x^2 + \frac{1}{n}} + |x|}\right|$$
   $$= \frac{1}{n}\,\frac{1}{\sqrt{x^2 + \frac{1}{n}} + |x|} \leq \frac{1}{n}\,\frac{1}{\sqrt{1/n}} = \frac{1}{\sqrt{n}}.$$

   In fact this upper bound is attained at $x = 0$, so that

   $$d(f_n, f) = \frac{1}{\sqrt{n}}.$$

   Hence $f_n \to f$ uniformly on $[-1,1]$.

   However, we can easily compute $f_n'(0) = 0$ for every n, while $f'(0)$ does not exist, so the derivative does not converge pointwise.

3. Textbook problems, 7.3: 2.

   Suppose $f_n \to f$ (pointwise) and the functions f_n all satisfy the Lipschitz condition $|f_n(x) - f_n(y) \le M|x - y|$ for some constant M independent of n. Prove that f also satisfies the same Lipschitz condition.

   Solution:

   For any two fixed points x and y, the sequence $|f_n(x) - f_n(y)|$ converges to $|f(x) - f(y)|$, since the absolute value function is continuous. By Theorem 2.3.2.b, since $|f_n(x) - f_n(y)| \le M|x - y|$ for every n, we also have $|f(x) - f(y)| \le M|x - y|$.

4. Textbook problem, 7.3: 5.

   If $\lim_{n\to\infty} f_n = f$ and the functions f_n are all monotone increasing, must f be monotone increasing? What happens if f_n are all strictly increasing?

   Solution:

   In either case, f must be monotone increasing. Let $x < y$ be any two points; then we know $f_n(x) \le f_n(y)$ for every n. So $f(x) \le f(y)$ again by Theorem 2.3.2.b, and hence f is monotone increasing.

   If f_n is strictly increasing, then $f_n(x) < f_n(y)$ for every n. In the limit we can only say that $f(x) \le f(y)$. Hence even if every f_n is strictly increasing, f may only be monotone increasing.

   As an example of the latter, consider $f_n(x) = \frac{x}{n}$. f_n converges to the zero function f pointwise (and uniformly on any compact interval), but f is not strictly increasing.

5. Prove that $f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$ converges uniformly to $f(x) = \cos x$ on $[-a, a]$ for any real a. However show that f_n does not converge uniformly to f on $\mathbb{R}$.

   Solution:

   Observe that $f_n(x) = T_{2n}(x)$, the $2n^{\text{th}}$ Taylor polynomial of f about $x_o = 0$. So by the Taylor remainder formula, we have

   $$|f_n(x) - f(x)| = |T_{2n}(x) - f(x)| = \left| \frac{f^{(2n+1)}(x^*)(x-0)^{2n+1}}{(2n+1)!} \right|,$$

   for some x^* between 0 and x. However, since $f^{(2n+1)}(x) = (-1)^{n+1} \sin x$ for every x, we know that $|f^{(2n+1)}(x^*)| \le 1$, so we can estimate

   $$|f_n(x) - f(x)| \le \frac{|x|^{2n+1}}{(2n+1)!}.$$

   Hence

   $$\sup_{x\in[-a,a]} |f_n(x) - f(x)| \le \frac{a^{2n+1}}{(2n+1)!}.$$

   We know that for any $a \in \mathbb{R}$, $\lim_{n\to\infty} \frac{a^{2n+1}}{(2n+1)!} = 0$. So $f_n \to f$ uniformly on $[-a, a]$.

However, f_n does not converge uniformly to f on all of $\mathbb{R}$. The reason is that

$$d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \geq \sup_{x\in\mathbb{R}} |f_n(x)| - 1.$$

Since $f_n(x)$ is a polynomial, it approaches $\pm\infty$ as $x \to \pm\infty$. On the other hand, the cosine is always between -1 and 1. So $d(f_n, f) = \infty$ for every n, and hence f_n cannot converge uniformly to f.

6. Let $\mathcal{F}([a, b])$ be the set of all bounded functions $f\colon [a, b] \to \mathbb{R}$. We write $d(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|$ for the uniform norm.

   We say that a function $H\colon \mathcal{F}([a, b]) \to \mathbb{R}$ is *uniformly continuous* on $\mathcal{F}$ if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any functions f and g with $d(f, g) < \delta$, we have $|H(f) - H(g)| < \varepsilon$.

   (a) Let $c \in [a, b]$ be any point, and let $E\colon \mathcal{F}[a, b] \to \mathbb{R}$ be the *evaluation function*, $E(f) = f(c)$. Show that E is a uniformly continuous function.

   Solution:

   This is easy once you know how to approach it:

$$|E(f) - E(g)| = |f(c) - g(c)| \leq \sup_{x\in[a,b]} |f(x) - g(x)| = d(f, g).$$

   So given any $\varepsilon > 0$, choose $\delta = \varepsilon$, and we're done.

   (b) Let $I\colon \Omega \subset \mathcal{F}[a, b] \to \mathbb{R}$ be the integral $I(f) = \int_a^b f(x)\,dx$. (Here Ω is the set of all *integrable* functions, which is a subset of the set of *all* bounded functions.) Prove that I is a uniformly continuous function. (This is similar to but slightly stronger than Theorem 7.3.3.)

   Solution:

   If f and g are both integrable, then so is $f - g$, and so is $|f - g|$. Therefore

$$\begin{aligned}
|I(f) - I(g)| &= \left| \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right| \\
&= \left| \int_a^b \big(f(x) - g(x)\big)\,dx \right| \\
&\leq \int_a^b |f(x) - g(x)|\,dx \\
&\leq \int_a^b \sup_{x\in[a,b]} |f(x) - g(x)|\,dx \\
&= \int_a^b d(f, g)\,dx \\
&= (b - a)d(f, g).
\end{aligned}$$

   So given any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{b-a}$, and we're done.

(c) Give an example of a function $D\colon \mathcal{F}[a,b] \to \mathbb{R}$ that is *not* continuous. (Hint: you could try composing a discontinuous function of real numbers with the evaluation function.)

Solution:

Following the hint, let c be any number in $[a,b]$ and let $D\colon \mathcal{F}[a,b] \to \mathbb{R}$ be

$$D(f) = \begin{cases} 0 & f(c) < 0 \\ 1 & f(c) \geq 0. \end{cases}$$

Then $D(f)$ is discontinuous, as seen for example with the constant functions $f_n(x) \equiv -\frac{1}{n}$ which converge uniformly to $f(x) \equiv 0$: we have $D(f_n) = 0$ for every n, while $D(f) = 1$.