Math 4310 Homework #11 Solutions

1. In Calculus, the "Limit Comparison Test" is a frequently used alternative to the Comparison Test. The Limit Comparison Test says that if $\sum a_n$ and $\sum b_n$ are two series with positive terms, and if $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $L \neq 0$ and $L \neq \infty$, then either $\sum a_n$ and $\sum b_n$ both converge or they both diverge.

   Prove the Limit Comparison Test.

   Solution:

   Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. First we prove that if $\sum b_n$ converges, then also a_n converges. Letting $\varepsilon = 1$ and using the definition of limit, we know that there is an N such that $n \geq N$ implies $\frac{a_n}{b_n} < L + 1$. Thus for $n \geq N$, we have $a_n < (L+1)b_n$. Now the first N terms do not affect convergence, and thus by the ordinary comparison test, we know that $\sum a_n$ converges.

   Similarly, if we assume $\sum a_n$ converges, then there is an N such that $n \geq N$ implies $\frac{b_n}{a_n} < \frac{1}{L} + 1$, so that $b_n < \left(\frac{1}{L} + 1\right) a_n$, and again the ordinary comparison test tells us that $\sum b_n$ converges.

2. Prove Theorem 7.2.3.

   (a) The Ratio Test: If $|x_{n+1}/x_n| < r$ for all sufficiently large n and some $r < 1$, then $\sum x_n$ converges absolutely. If $|x_{n+1}/x_n| \geq 1$ for all sufficiently large n, then $\sum x_n$ diverges.

   Solution:

   There is some N such that $|x_{n+1}/x_n| < r$ for all $n \geq N$. Hence

$$|x_{N+1}| < r|x_N|$$
$$|x_{N+2}| < r|x_{N+1}| < r^2|x_N|$$
$$|x_{N+3}| < r|x_{N+2}| < r^3|x_N|,$$

   etc. In general we have inductively that $|x_{N+k}| \leq r^k|x_N|$ for every integer $k \geq 0$, or rewriting slightly (with $j = N + k$) we have

$$|x_j| \leq r^{j-N}|x_N| = \frac{|x_N|}{r^N} r^j.$$

   Now the term $\frac{|x_N|}{r^N}$ is a constant, which does not affect convergence of the series by which it is multiplied. Hence the comparison test (and the fact that the geometric series converges for $r < 1$) implies that $\sum x_n$ converges absolutely.

   The second part is easy: if $|x_{n+1}/x_n| \geq 1$ for all n sufficiently large, then $|x_k| \geq |x_N|$ for all $k \geq N$, so that the terms of the series cannot converge to zero (and hence the series must diverge by the n^{th} term test).

   (b) The Root Test: If $|x_n|^{1/n} < r$ for all sufficiently large n and some $r < 1$, then $\sum x_n$ converges absolutely.

   Solution:

   Again, there is some N such that $|x_n|^{1/n} < r$ for all $n \geq N$. Hence for $n \geq N$, we have $|x_n| < r^n$. By the comparison test, we see that $\sum x_n$ converges absolutely.

(Hint for both: they're easy if you use the Comparison Test.)

3. (a) Prove that the Ratio Test is a consequence of the Root Test; in other words, if $|x_{n+1}/x_n| < r$ for all sufficiently large n and some $r < 1$, then also $|x_n|^{1/n} < R$ for all sufficiently large n and some $R < 1$.

(Hint: of course you can just say for some N and r that $|x_{N+1}| < r|x_N|$, and $|x_{N+2}| < r^2|x_N|$, etc. Prove that eventually $|x_{N+k}|^{1/(N+k)}$ is less than 1. Note that your R will probably *not* be the same as r, and that "sufficiently large" for the Ratio Test will probably not be the same as "sufficiently large" for the Root Test.)

Solution:

As in the hint, we see inductively that $|x_{N+k}| \leq r^k|x_N|$, so that $|x_j| \leq r^{j-N}|x_N|$ for all $j \geq N$.

Therefore we have

$$|x_j|^{1/j} \leq r\left(\frac{|x_N|}{r^N}\right)^{1/j}$$

for all j. Now since $\frac{|x_N|}{r^N}$ is a fixed constant (independent of j), we know that $\lim_{j\to\infty}\left(\frac{|x_N|}{r^N}\right)^{1/j} = 1$, so that

$$\lim_{j\to\infty} r\left(\frac{|x_N|}{r^N}\right)^{1/j} = r.$$

Hence (using the definition of limit, with $\varepsilon = (1-r)/2$), there is some integer M such that $j \geq M$ implies

$$r\left(\frac{|x_N|}{r^N}\right)^{1/j} < 1 - \frac{r}{2}.$$

Denoting $R = 1 - \frac{r}{2}$, we find that

$$|x_j|^{1/j} < R \quad \text{for } j \geq M.$$

Hence since $R < 1$, the series satisfies the conditions of the root test.

(b) For the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}},$$

show that the Root Test gives convergence, but the Ratio Test fails to tell you anything.

Solution:

Using the root test, we get

$$\begin{aligned}\lim_{n\to\infty}|a_n|^{1/n} &= \lim_{n\to\infty}\left(\frac{1}{2^{n+(-1)^n}}\right)^{1/n} \\ &= \lim_{n\to\infty}\frac{1}{2^{1+(-1)^n/n}} \\ &= \frac{1}{2}.\end{aligned}$$

Since this value $r = \frac{1}{2}$ is less than 1, the series converges by the root test.

Using the ratio test, we get

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+(-1)^n}}{2^{n+1+(-1)^{n+1}}} = 2^{(-1)^n - 1 - (-1)^{n+1}} = 2^{2(-1)^n - 1}.$$

If n is even, we have $\frac{a_{n+1}}{a_n} = \frac{1}{2}$, while if n is odd, we have $\frac{a_{n+1}}{a_n} = 2$. Hence it is not true that $\frac{a_{n+1}}{a_n}$ is eventually less than 1, nor is it eventually greater than 1; so the ratio test cannot be used for either convergence or divergence.

4. Let $\mathcal{F}[a,b]$ be the set of all bounded functions $f\colon [a,b] \to \mathbb{R}$. The "$C^0$ norm" on $\mathcal{F}[a,b]$ is the distance

$$d(f,g) = \sup_{a \le x \le b} |f(x) - g(x)|.$$

(a) Prove that $d(f,g) = 0$ if and only if $f(x) = g(x)$ for all $x \in [a,b]$.

Solution:

If $f(x) = g(x)$ for all $x \in [a,b]$, we know $|f(x) - g(x)| = 0$ for all $x \in [a,b]$, and thus $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| = 0$.

On the other hand, if $d(f,g) = 0$, then $\sup_{x \in [a,b]} |f(x) - g(x)| = 0$, so that for every $x \in [a,b]$ we have $|f(x) - g(x)| = 0$, which implies that for every $x \in [a,b]$ that $f(x) = g(x)$.

(b) Prove the triangle inequality:

$$d(f,h) \le d(f,g) + d(g,h).$$

Do this carefully!

Solution:

For any fixed $x \in [a,b]$, we have

$$\begin{aligned} |f(x) - h(x)| &\le |f(x) - g(x)| + |g(x) - h(x)| \\ &\le \sup_{x \in [a,b]} |f(x) - g(x)| + \sup_{x \in [a,b]} |g(x) - h(x)| \\ &= d(f,g) + d(g,h). \end{aligned}$$

Since this is true for every $x \in [a,b]$, we know

$$\sup_{x \in [a,b]} |f(x) - h(x)| \le d(f,g) + d(g,h),$$

or in other words that $d(f,h) \le d(f,g) + d(g,h)$.