Math 4310 Homework #10 Solutions

1. Textbook exercise, 5.4.6: 2

   Suppose $f'(x_o) = 0, f''(x_o) = 0, \cdots, f^{(n-1)}(x_o) = 0$ and $f^{(n)}(x_o) > 0$ for a C^n function f. Prove that f has a local minimum at x_o if n is even and that x_o is neither a local maximum nor a local minimum if n is odd.

   Solution:

   The n^{th} Taylor polynomial centered at x_o is

   $$T_n(x) = f(x_o) + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n.$$

   By Taylor's theorem, we have

   $$\lim_{x \to x_o} \frac{f(x) - T_n(x)}{(x - x_o)^n} = 0.$$

   Plugging in the formula for T_n and simplifying a bit, we get

   $$\lim_{x \to x_o} \frac{f(x) - f(x_o)}{(x - x_o)^n} = \frac{f^{(n)}(x_o)}{n!}.$$

   Let $\varepsilon = \frac{f^{(n)}(x_o)}{n!}$, and choose $\delta > 0$ so that $|x - x_o| < \delta$ implies

   $$\left| \frac{f(x) - f(x_o)}{(x - x_o)^n} - \frac{f^{(n)}(x_o)}{n!} \right| < \varepsilon;$$

   then $|x - x_o| < \delta$ implies that

   $$\frac{f(x) - f(x_o)}{(x - x_o)^n} > 0.$$

   Now we consider the cases separately. If n is even, then $(x - x_o)^n$ is positive whenever $x \neq x_o$, so that if $0 < |x - x_o| < \delta$ we have $f(x) - f(x_o) > 0$. Hence x_o is a strict local minimum.

   Now if n is odd, then $(x - x_o)$ is positive for $x > x_o$ and negative for $x < x_o$. Hence for $x_o < x < x_o + \delta$ we have $f(x) > f(x_o)$, while for $x_o - \delta < x < x_o$ we have $f(x) < f(x_o)$. So the function is strictly increasing at x_o, and in particular it is neither a local maximum nor a local minimum.

2. Textbook exercise, 5.4.6: 3

   If f is C^2 on an interval prove that

   $$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).$$

Solution: There are two ways to prove this: one is to use L'Hôpital's Rule, and the other is to use Taylor's Theorem. Of course, since L'Hôpital's Rule is derived from Taylor's Theorem, the two approaches are equivalent.

From Taylor's Theorem, we have

$$\lim_{h\to 0}\frac{f(x+h)-T_n(x+h)}{h^2}=0 \quad\text{and}\quad \lim_{h\to 0}\frac{f(x-h)-T_n(x-h)}{2}=0.$$

Hence

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}\\
&\qquad=\lim_{h\to 0}\tfrac{[f(x+h)-T_n(x+h)]+T_n(x+h)-2f(x)+[f(x-h)-T_n(x-h)]+T_n(x-h)}{h^2}\\
&\qquad=\lim_{h\to 0}\tfrac{f(x+h)-T_n(x+h)}{h^2}+\lim_{h\to 0}\frac{T_n(x+h)-2f(x)+T_n(x-h)}{h^2}\\
&\qquad\qquad+\lim_{h\to 0}\frac{f(x-h)-T_n(x-h)}{h^2}\\
&\qquad=\lim_{h\to 0}\frac{T_n(x+h)-2f(x)+T_n(x-h)}{h^2}.
\end{aligned}$$

Now $T_n(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)$ and $T_n(x-h)=f(x)-hf'(x)+\frac{h^2}{2}f''(x)$, so that

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}&=\lim_{h\to 0}\frac{T_n(x+h)-2f(x)+T_n(x-h)}{h^2}\\
&=\lim_{h\to 0}\frac{f(x)+hf'(x)+\frac{h^2}{2}f''(x)-2f(x)+f(x)-hf'(x)+\frac{h^2}{2}f''(x)}{h^2}\\
&=\lim_{h\to 0}\frac{h^2f''(x)}{h^2}\\
&=f''(x).
\end{aligned}$$

3. Textbook exercise, 5.4.6: 4

   Let f and g be C^3 functions with $f(x_o)=g(x_o)=0$ but $g'(x_o)\neq 0$. Show that the derivative of f/g is continuous at x_o.

   Solution:

   By L'Hôpital's Rule, we have

   $$\lim_{x\to x_o}\frac{f(x)}{g(x)}=\frac{f'(x_o)}{g'(x_o)},$$

   and so the discontinuity of f/g is removable. Now from the definition, the derivative of f/g at x_o is

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_o) &= \lim_{x\to x_o} \frac{\frac{f(x)}{g(x)} - \frac{f'(x_o)}{g'(x_o)}}{x - x_o} \\
&= \lim_{x\to x_o} \frac{g'(x_o)f(x) - f'(x_o)g(x)}{g(x)g'(x_o)(x-x_o)} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to x_o} \frac{g'(x_o)f'(x) - f'(x_o)g'(x)}{g(x)g'(x_o) + g'(x)g'(x_o)(x-x_o)} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to x_o} \frac{g'(x_o)f''(x) - f'(x_o)g''(x)}{2g'(x)g'(x_o) + g''(x)g'(x_o)(x-x_o)} \\
&= \frac{g'(x_o)f''(x_o) - f'(x_o)g''(x_o)}{2g'(x_o)^2}.
\end{aligned}$$

We have proved that f/g is differentiable at x_o, but now we need to show the derivative is continuous, i.e., that $\lim_{x\to x_o}(f/g)'(x) = (f/g)'(x_o)$. Again we use L'Hôpital's Rule, along with the quotient rule for f/g (which works for all x near x_o but not equal to x_o, since $g'(x_o) \neq 0$ implies that $g(x)$ is not equal to zero for x near x_o).

$$\begin{aligned}
\lim_{x\to x_o} \left(\frac{f}{g}\right)'(x) &= \lim_{x\to x_o} \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to x_o} \frac{g'(x)f'(x) + g(x)f''(x) - f'(x)g'(x) - f(x)g''(x)}{2g(x)g'(x)} \\
&= \lim_{x\to x_o} \frac{g(x)f''(x) - f(x)g''(x)}{2g(x)g'(x)} \left(= \frac{0}{0}\right) \\
&= \lim_{x\to x_o} \frac{g'(x)f''(x) + g(x)f'''(x) - f'(x)g''(x) - f(x)g'''(x)}{2g'(x)^2 + 2g(x)g''(x)} \\
&= \frac{g'(x_o)f''(x_o) - f'(x_o)g''(x_o)}{2g'(x_o)^2}.
\end{aligned}$$

4. Textbook exercise, 6.1.5: 4

   Prove the integral mean value theorem: if f is continuous on $[a,b]$, then there exists $y \in (a,b)$ such that $\int_a^b f(x)\,dx = (b-a)f(y)$.

   Solution: Since f is continuous on $[a,b]$, it attains its maximum and minimum: there are points c and d such that $f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$. Hence we know

   $$(b-a)f(c) \le \int_a^b f(x)\,dx \le (b-a)f(d).$$

   As a result, the number $\frac{\int_a^b f(x)\,dx}{b-a}$ is between $f(c)$ and $f(d)$. Hence by the Intermediate Value Theorem, there is some y between c and d such that

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(y).$$

5. Textbook exercise, 6.1.5: 5

   Let g be continuous on $[a,b]$, and let $f(x) = \int_a^x (x-t)g(t)\,dt$. Prove that f is a solution of the differential equation $f'' = g$ and the initial conditions $f(a) = f'(a) = 0$.

   Solution: We can rewrite

   $$f(x) = x\int_a^x g(t)\,dt - \int_a^x tg(t)\,dt \equiv xp(x) - q(x),$$

   and so by the product rule and the fundamental theorem of calculus, we have

   $$\begin{aligned} f'(x) &= xp'(x) + p(x) - q'(x) \\ &= xg(x) + \int_a^x g(t)\,dt - xg(x) \\ &= \int_a^x g(t)\,dt. \end{aligned}$$

   Furthermore, again by the fundamental theorem of calculus, we have

   $$f''(x) = g(x).$$

   Finally, we obviously have $f(a) = \int_a^a (x-t)g(t)\,dt = 0$ and $f'(a) = \int_a^a g(t)\,dt = 0$.

6. Textbook exercise, 6.1.5: 8

   Let f be a C^1 function on the line, and let $g(x) = \int_0^1 f(xy)y^2\,dy$. Prove that g is a C^1 function and establish a formula for $g'(x)$ in terms of f.

   Solution:

   We compute from the definition:

   $$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{1}{h}\left[\int_0^1 f(xy+hy)y^2\,dy - \int_0^1 f(xy)y^2\,dy\right] \\ &= \frac{1}{h}\int_0^1 y^2[f(xy+hy) - f(xy)]\,dy. \end{aligned}$$

   Now we can compute for each fixed y that

   $$\begin{aligned} \lim_{h\to 0} \frac{f(xy+hy)-f(xy)}{h} &= \lim_{h\to 0} y\,\frac{f(xy+hy)-f(xy)}{hy} \\ &= y\lim_{j\to 0}\frac{f(xy+j)-f(xy)}{j} = yf'(xy). \end{aligned}$$

This gives us our candidate for what's inside the integral:

$$\begin{aligned}
&\left|\frac{g(x+h)-g(x)}{h} - \int_0^1 f'(xy)y^3\,dy\right| \\
&\qquad = \left|\int_0^1 y^2\left[\frac{f(xy+hy)-f(xy)}{h} - f'(xy)y\right]dy\right| \\
&\qquad \le \int_0^1 y^2\left|\frac{f(xy+hy)-f(xy)-yf'(xy)h}{h}\right|dy \\
&\qquad \le \sup_{0\le y\le 1}\left|\frac{f(xy+hy)-f(xy)-hyf'(xy)}{h}\right|.
\end{aligned}$$

Now for each fixed x, y, and h, there is some number $q(x,y,h) \in [xy, xy+hy]$ such that

$$f(xy+hy)-f(xy) = hyf'(q).$$

Hence we have

$$\frac{f(xy+hy)-f(xy)-hyf'(xy)}{h} = yf'(q) - yf'(xy).$$

Now since f' is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$, and hence for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|u-v| < \delta$ implies $|f'(u)-f'(v)| < \varepsilon$.

So let $\varepsilon > 0$ be any number, and choose δ as above. Then $|h| < \delta$ and $0 \le y \le 1$ implies $|hy| < \delta$. So let x, y, and h be any numbers with $0 \le y \le 1$ and $|h| < \delta$; then

$$\begin{aligned}
\left|\frac{f(xy+hy)-f(xy)}{h} - yf'(xy)\right| &= |yf'(q) - yf'(xy)| \\
&\le y|f'(q) - f'(xy)| \\
&< y\varepsilon <= \varepsilon.
\end{aligned}$$

We conclude that for $|h| < \delta$, we have

$$\sup_{0\le y\le 1}\left|\frac{f(xy+hy)-f(xy)-hyf'(xy)}{h}\right| < \varepsilon.$$

Hence using the estimate above, we conclude that $|h| < \delta$ implies

$$\left|\frac{g(x+h)-g(x)}{h} - \int_0^1 f'(xy)y^3\,dy\right| < \varepsilon.$$

Hence by the definition of the derivative, we have

$$g'(x) = \int_0^1 f'(xy)y^3\,dy.$$

To prove g' is continuous, we use the fact that f' is continuous. Let x be any number. Then since f' is uniformly continuous on $[0, x]$, we know that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|u - v| < \delta$ implies $|f'(u) - f'(v)| < \varepsilon$. So if $|yh| < \delta$, then $|f'(y(x + h)) - f'(yx)| < \varepsilon$. So $|h| < \delta$ implies $|yh| < \delta$, which leads to

$$|g'(x + h) - g'(x)| \leq \int_0^1 y^3 |f'(xy + yh) - f'(xy)| \, dy < \int_0^1 y^3 \varepsilon \, dy < \varepsilon.$$

So g is indeed continuous at x.