Math 4310 Final Exam Solutions

1. (20 points) Suppose $f_n \colon D \to \mathbb{R}$ is a sequence of functions which is uniformly continuous on D. If f_n converges uniformly to $f \colon D \to \mathbb{R}$, prove directly from the definitions that f is also uniformly continuous on D.

   Solution:

   Recall the definition of uniform continuity: for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x, y \in D$ with $|x - y| < \delta$, we have

   $$|f(x) - f(y)| < \varepsilon.$$

   To estimate $|f(x) - f(y)|$, we use the old add-and-subtract trick:

   $$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq d(f, f_n) + |f_n(x) - f_n(y)| + d(f_n, f) \\ &= 2d(f, f_n) + |f_n(x) - f_n(y)|. \end{aligned}$$

   Since $\lim_{n\to\infty} d(f, f_n) = 0$, for any $\varepsilon > 0$ we can find an n such that $d(f, f_n) < \frac{\varepsilon}{3}$. For this n, f_n is uniformly continuous, so there is a $\delta > 0$ such that for any $x, y \in D$ with $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$. So for this n and this δ, we have $|x - y| < \delta$ implying

   $$|f(x) - f(y)| \leq 2d(f, f_n) + |f_n(x) - f_n(y)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

   So f is uniformly continuous, as desired.

2. (15 points) State and prove L'Hôpital's Rule, using any version of Taylor's theorem. Be clear about the assumptions you're making on the functions involved.

   Solution: One version is that if f and g are C^n functions with $f(a) = f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, and $g(a) = g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = 0$ and $g^{(n)}(a) \neq 0$, then

   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

   To prove this formula, we use Taylor's theorem in the form

   $$\lim_{x\to a} \frac{f(x) - T_n(x)}{(x-a)^n} = 0,$$

   which is valid as long as f is C^n. Then by the assumptions on the derivatives, $T_n^f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n$ for f and $T_n^g(x) = \frac{g^{(n)}(a)}{n!}(x-a)^n$ for g. So using our favorite trick, we have

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-T_n^f(x)+T_n^f(x)}{g(x)-T_n^g(x)+T_n^g(x)}\\
&= \lim_{x\to a}\frac{\frac{f(x)-T_n^f(x)}{(x-a)^n}+\frac{T_n^f(x)}{(x-a)^n}}{\frac{g(x)-T_n^g(x)}{(x-a)^n}+\frac{T_n^g(x)}{(x-a)^n}}\\
&= \lim_{x\to a}\frac{0+\frac{f^{(n)}(a)}{n!}}{0+\frac{g^{(n)}(a)}{n!}}\\
&= \frac{f^{(n)}(a)}{g^{(n)}(a)}.
\end{aligned}$$

3. (20 points) Suppose $|f| \le M$ on $[a,b]$. Prove that if f is integrable on $[a,b]$, then f^2 is also integrable, from the definition. (Hint: first show that $|f^2(x)-f^2(y)| \le 2M|f(x)-f(y)|$ for every $x, y \in [a,b]$.)

   Solution:

   We know f^2 is integrable if and only if $\sup_P S^-(f^2,P) = \inf_P S^+(f^2,P)$. This will follow if for every $\varepsilon > 0$, there is a partition P such that $S^+(f^2,P) - S^-(f^2,P) < \varepsilon$.

   As suggested in the hint, we have

$$\begin{aligned}
|f^2(x)-f^2(y)| &= |\big(f(x)+f(y)\big)\big(f(x)-f(y)\big)|\\
&\le |f(x)+f(y)||f(x)-f(y)|\\
&\le \big(|f(x)|+|f(y)|\big)|f(x)-f(y)|\\
&\le 2M|f(x)-f(y)|.
\end{aligned}$$

   This tells us in particular that

$$f^2(x)-f^2(y) \le 2M\big(\sup_{x\in D} f(x) - \inf_{y\in D} f(y)\big)$$

   for any interval D and for every x and y in D. Thus taking first the supremum over x, then the infimum over $y \in D$, we obtain that

$$\sup_{x\in D} f^2(x) - \inf_{y\in D} f^2(y) \le 2M\big(\sup_{x\in D} f(x) - \inf_{y\in D} f(y)\big),$$

   for any interval D.

   Now for any partition P, we have

$$\begin{aligned}
S^+(f^2,P)-S^-(f^2,P) &= \sum_{k=1}^{n}\Big(\sup_{x_{k-1}\le x\le x_k} f^2(x) - \inf_{x_{k-1}\le y\le x_k} f^2(y)\Big)(x_k-x_{k-1})\\
&\le \sum_{k=1}^{n} 2M\Big(\sup_{x_{k-1}\le x\le x_k} f(x) - \inf_{x_{k-1}\le y\le x_k} f(y)\Big)(x_k-x_{k-1})\\
&\le 2M\big(S^+(f,P)-S^-(f,P)\big).
\end{aligned}$$

Let $\varepsilon > 0$ be any number, and choose a partition P so that

$$S^+(f, P) - S^-(f, P) < \frac{\varepsilon}{2M}.$$

Then by the inequality above, we know that

$$S^+(f^2, P) - S^-(f^2, P) < \varepsilon.$$

Since this can be done for any $\varepsilon > 0$, we conclude f^2 is integrable.

4. (20 points) Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k} = \ln 2$, using anything we proved in class. (Hint: it's a power series at a particular value of x.)

   Solution: Clearly we can identify $\frac{1}{2^k} = x^k$ when $x = \frac{1}{2}$. So we are dealing with the function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. Since $\lim_{k\to\infty}(1/k)^{1/k} = 1$, the radius of convergence of this power series is $R = 1$. Hence in particular it converges uniformly on the interval $[-\frac{3}{4}, \frac{3}{4}]$, by a theorem from class. (I chose $\frac{3}{4}$ since it's between $x = \frac{1}{2}$, the value we're using, and $R = 1$, the radius of convergence.)

   Now a power series can be differentiated term by term inside its radius of convergence (by another theorem from class), so that for any x in $(-1, 1)$ we have

   $$f'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k} = \sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \cdots .$$

   We instantly recognize this as the geometric series, which converges for $x \in (-1, 1)$ to $f'(x) = \frac{1}{1-x}$.

   Now by the Fundamental Theorem of Calculus,

   $$f(x) - f(0) = \int_0^x f'(t)\,dt = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-t)\Big|_0^x = \ln 1 - \ln(1-x).$$

   We obviously have $f(0) = 0$ from the original power series, and $f(\frac{1}{2})$ is the term we want. So

   $$\sum_{k=1}^{\infty} \frac{1}{k2^k} = f(\tfrac{1}{2}) = \int_0^{1/2} f'(t)\,dt = -\ln(1 - 1/2) = \ln 2.$$

5. (25 points) True or false. If true, state in one sentence the results proved in class that imply it. If false, give an explicit counterexample. (Half credit for the correct answer, half credit for the justification.)

   (a) (8 points) If f_n converges to f uniformly on some domain, then f_n' converges to f' pointwise on that domain.

   Solution:

   False. For an example, see the functions from homework, $f_n(x) = \sqrt{1/n + x^2}$ which converge uniformly to $f(x) = |x|$ on the interval $[-1, 1]$. We see $f_n'(0) = 0$ for every n, while $f'(0)$ does not exist.

(b) (9 points) If f_n converges pointwise to f, then $\int_a^b f_n(x)\,dx$ converges to $\int_a^b f(x)\,dx$.

Solution:

False. Pointwise convergence is not enough to guarantee that the integrals converge. For an example, consider the functions from class, given by the triangle graphs: f_n is the line from $(0,0)$ to $(\frac{1}{n}, n)$, then the line from $(\frac{1}{n}, n)$ to $(\frac{2}{n}, 0)$, then the zero function elsewhere. We know that f_n converges pointwise to the zero function on $[0,1]$, while $\int_0^1 f_n(x)\,dx = 1$ for every n, so the integrals do not converge to zero.

(c) (8 points) If $\sum_{n=1}^\infty a_n$ is a series of positive terms, and if $\frac{a_{n+1}}{a_n} < 1$ for every n, then $\sum_{n=1}^\infty a_n$ converges.

Solution:

False. This is almost the ratio test, but the fact that the ratio can approach 1 means that it does not actually give any information. In fact notice that the question just says that the terms are decreasing: $a_{n+1} < a_n$ for every n. Clearly that is not enough to ensure convergence of the series. For an explicit example, consider the harmonic series $a_n = \frac{1}{n}$. Then $a_{n+1} < a_n$ for every n, yet the series does not converge.