Math 4310 Second Exam Solutions

1. (15 points) Prove directly from the definition that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 6)$.

   Solution:

   First compute $f(x) - f(x_o)$:

   $$f(x) - f(x_o) = \frac{1}{x} - \frac{1}{x_o} = \frac{x_o - x}{x x_o}.$$

   Now since $1 \leq x$, we know $\frac{1}{x} \leq 1$; similarly $\frac{1}{x_o} \leq 1$. Hence

   $$|f(x) - f(x_o)| = \frac{|x - x_o|}{x x_o} \leq |x - x_o|$$

   for all x, x_o in $[1, 6)$.

   So let $\varepsilon > 0$ be any number, and choose $\delta = \varepsilon$. Then $|x - x_o| < \delta$ and $x, x_o \in [1, 6)$ implies

   $$|f(x) - f(x_o)| \leq |x - x_o| < \delta = \varepsilon,$$

   which is the definition of uniform continuity.

2. (20 points) Prove, using any theorems we proved in class, that there is exactly one number x such that $x^5 + 3x + 1 = 0$. (First prove there is at least one; then prove there cannot be more than one.)

   Solution:

   First, the function $f(x) = x^5 + 3x + 1$ is a polynomial, so it is differentiable and continuous. Since it is continuous everywhere, and since $f(-1) = -1$ and $f(0) = 1$, there is (by the intermediate value theorem) a point $x_o \in (-1, 0)$ such that $f(x_o) = 0$.

   Secondly, we know by the calculus rules that $f'(x) = 5x^4 + 3 > 0$ for all x. Hence f is strictly increasing, so that f must be one-to-one. So there cannot be any other x with $f(x) = 0$.

3. (15 points) Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Prove Rolle's Theorem, a special case of the Mean Value Theorem: "If $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that $f'(c) = 0$."

   (You may use any theorems proved in class, except of course the Mean Value Theorem.)

   Solution:

   Since f is continuous on the compact set $[a, b]$, it attains its maximum M and minimum m. If $M > f(a)$ then there is a $c \in (a, b)$ with $f(c) = M$. Then c is a local maximum, so $f'(c) = 0$. If $m < f(a)$, then there is a $d \in (a, b)$ with $f(d) = m$; then d is a local minimum, so $f'(d) = 0$.

   The only other possibility is that $m = M = f(a)$, which implies that f is constant. In this case, $f'(x) = 0$ for all $x \in (a, b)$.

4. (20 points) Prove the Fundamental Theorem of Calculus (part two): if f is continuously differentiable, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

(You may use any theorems proved in class, except of course this one.)

Solution:

Since f is continuously differentiable, f' is a continuous function and hence it is integrable. Now take any partition P of $[a,b]$, with $P = \{x_0, x_1, \cdots, x_n\}$. For each k, choose (by the mean value theorem) a point $q_k \in (x_{k-1}, x_k)$ such that

$$f'(q_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

Consider the sum

$$\sum_{k=1}^{n} f'(q_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(b) - f(a).$$

Since $f'(q_k) \leq \sup_{x_{k-1} \leq x \leq x_k} f(x) = M_k$ for every k, we know $f(b) - f(a) \leq S^+(f,P)$. Similarly we have $S^-(f,P) \leq f(b) - f(a)$.

Now $f(b) - f(a)$ is a lower bound for the upper Riemann sums $S^+(f,P)$ over all partitions; hence

$$f(b) - f(a) \leq U(f) = \int_a^b f'(x)\,dx.$$

Similarly it is an upper bound for the lower Riemann sums, and hence

$$f(b) - f(a) \geq L(f) = \int_a^b f'(x)\,dx.$$

The only way both inequalities can be true is if

$$f(b) - f(a) = \int_a^b f'(x)\,dx,$$

which is what we wanted to prove.

5. (30 points) For each of the following, indicate whether the statement is true or or false. If true, write in one sentence what theorem(s) imply it. If false, give an explicit counterexample.

   (a) If f is differentiable on $[a,b]$, then f is integrable on $[a,b]$.

   Solution:

   True. f is differentiable, so f is continuous; this implies f is integrable.

(b) If $|f|$ is integrable on $[a, b]$, then f is integrable on $[a, b]$.

Solution:

False. Let

$$f(x) = \begin{cases} 1 & x \text{ is rational,} \\ -1 & x \text{ is irrational.} \end{cases}$$

Then f is not integrable, since $S^+(f, P) = b - a$ and $S^-(f, P) = -(b - a)$ for every partition P, and thus $U(f) = b - a$ and $L(f) = -(b - a)$. However, $|f(x)| = 1$ for all x, and hence $|f|$ is continuous (thus integrable).

(c) For any integrable function f on $[a, b]$,

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

Solution:

False. It would be true if f were continuous, by the fundamental theorem of calculus (part one). Consider the example from the homework:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then f is integrable since it is monotone. However, if $[a, b] = [-1, 1]$, then

$$F(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0, \\ x & \text{if } 0 \leq x \leq 1. \end{cases}$$

So F is not differentiable at $x = 0$.