Math 4310 First Exam Solutions

1. (20 points) Prove that the set of all real numbers in the interval $[0,1]$ with decimal expansions containing only the digits 3 and 5 is uncountable.

   Solution: The basic technique is just to use Cantor's diagonalization argument. Let S be the set
$$S = \{0.a_1a_2a_3\cdots \mid a_k \in \{3,5\} \text{ for all } k\}.$$
   Take any countable subset of S, and call it $x_1, x_2, x_3, \ldots$: Write each element as $x_j = 0.a_{j1}a_{j2}a_{j3}\cdots$, and then construct a matrix
$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \end{aligned}$$
   Now define $y = 0.b_1b_2b_3\cdots$, where
$$b_j = \begin{cases} 3 & \text{if } a_{jj} = 5 \\ 5 & \text{if } a_{jj} = 3 \end{cases}.$$
   Then y is a real number, and $y \neq x_j$ for every j, since for every j the j^{th} decimal place of y is different from the j^{th} decimal place of x_j.

   So for every countable subset E of S, we can find a $y \notin E$; hence S cannot be countable.

2. (15 points) What are the limit points of the sequence $x_k = \cos(\pi k/2) + \sin(\pi k/2)$, $k \in \mathbb{N}$? What are the limit points of the set $\{\cos(\pi k/2) + \sin(\pi k/2) \mid k \in \mathbb{N}\}$? Explain your answers, using anything we proved in class.

   Solution: The terms of the sequence are
$$1, -1, -1, 1, 1, -1, -1, 1, 1, \cdots.$$
   The limit points of the sequence are 1 and -1 (since there are obviously subsequences converging to each of these, and the terms cannot approach any other number).

   The set is $\{1, -1\}$, which has no limit points (no finite set ever has limit points, as we discussed in class).

3. (15 points) If $\lim_{k\to\infty} x_k = x$, prove that (x_k) is a Cauchy sequence, directly from the definitions.

   Solution: Let $\varepsilon > 0$ be any tolerance. By definition of limit, there is an integer $N \in \mathbb{N}$ such that whenever $k \geq N$, we have $|x_k - x| < \varepsilon/2$.

Hence as long as $j \geq N$ and $k \geq N$, we will have

$$\begin{aligned} |x_j - x_k| &= |x_j - x + x - x_k| \\ &\leq |x_j - x| + |x_k - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

4. (15 points) Suppose U is an open set in $\mathbb{R}$ which is bounded above. Prove that $\sup U$ is not in U, directly from the definitions.

   Solution: We know every set which is bounded above has a supremum; let us write $M = \sup U$. Assume, to get a contradiction, that M is in U. Then since U is open, there is some open interval (a, b) containing M and contained in U. Let $x = \frac{M+b}{2}$. Then $x \in (a, b)$, so $x \in U$; thus $x \leq M$ since M is an upper bound of U. However $x > M$ by construction, and hence we have a contradiction. Thus $M \notin U$.

5. (35 points) Let us say that y is a *schmimit* of a sequence $x_1, x_2, \cdots$ if there exists an $\varepsilon > 0$ such that, for every positive integer N, there is some $k \geq N$ with

$$|y - x_k| < \varepsilon.$$

   (a) (5 points) What is the difference between limit and schmimit?

   Solution: The number y is a limit of the sequence $x_1, x_2, x_3, \cdots$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $k \geq N$, we have $|y - x_k| < \varepsilon$.

   For schmimit, we have switched all of the qualifiers; we only need one $\varepsilon > 0$ for the tolerance, rather than arbitrarily small ε. In addition, instead of requiring that eventually all terms are within ε of the limit, we require only that infinitely many terms are within ε of the schmimit.

   Practically, any sequence with a limit point has all points $y \in \mathbb{R}$ as schmimit, while any sequence with no limit points will also have no schmimit points.

   (b) (10 points) Prove that if (x_k) converges to some x, then every $y \in \mathbb{R}$ is a schmimit of (x_k).

   Solution: Let $y \in \mathbb{R}$ be any number. We want $|y - x_k| < \varepsilon$ for some ε; we can bound the left side by

$$|y - x_k| \leq |y - x| + |x_k - x|.$$

   Now choose $\varepsilon = |y - x| + 1$. Since $\lim_{k\to\infty} x_k = x$, there is some integer N such that for every $k \geq N$, we have $|x_k - x| < 1$. Hence for every $k \geq N$, we have

$$|y - x_k| \leq |y - x| + |x_k - x| < |y - x| + 1 = \varepsilon.$$

   In particular, for every $N_1 \in \mathbb{N}$, we can choose $k \geq \max\{N_1, N\}$ and we have $|y - x_k| < \varepsilon$. Hence y is a schmimit.

(c) (10 points) What does it mean to say that y is not a schmimit of (x_k)? (Find the negation and simplify as much as possible.)

Solution: The number y is not a schmimit of (x_k) if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $k \geq N$, we have $|y - x_k| \geq \varepsilon$.

(d) (10 points) Find a sequence that has no schmimits. (Give an explicit formula for x_k and prove that for for every y, the number y is not a schmimit of (x_k).)

Solution: Any unbounded sequence with no limit points would work. For example, we can choose $x_k = k$. (Note: not every unbounded sequence would work. For example if $x_k = \frac{1}{k}$ when k is odd and $x_k = k$ when k is even, then x_k has all points as schmimits. It is essential that no subsequence converges in order to have no schmimits.)

So for example when $x_k = k$, let $y \in \mathbb{R}$. We will prove y is not a schmimit, i.e., we will show that for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that for every $k \geq N$, we have $|y - k| \geq \varepsilon$.

So let $\varepsilon > 0$. Choose $N \geq y + \varepsilon$. Then in particular $N > y$, so that for every $k \geq N$ we also have $k > y$. Thus whenever $k \geq N$, we have

$$|y - k| = k - y \geq N - y \geq \varepsilon,$$

as desired.