

Review Sheet for Final Exam

Mathematics 2300

December 19, 2006

The final exam will be cumulative. However, 3/5 of the final exam will be on Chapter 11, with 2/5 devoted to things covered in the rest of the course.
No calculators of any kind will be allowed.

Formulas you will be given on the exam (if they are needed):

- The transformation formulas $x' = x \cos \theta + y \sin \theta$ and $y' = -x \sin \theta + y \cos \theta$, as well as $x = x' \cos \theta - y' \sin \theta$ and $y = x' \sin \theta + y' \cos \theta$.
- The formulas relating A' and C' to A , B , and C when a conic section is rotated to remove the xy -term.

Formulas to remember:

- The polar-rectangular transformation formulas:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

- The most common polar graphs:
 - $r = c$ (circle of radius c centered at the origin)
 - $\theta = c$ (line of slope $m = \tan c$ through the origin)
 - $r = \pm c \cos \theta$ (circle of radius $c/2$ passing through the origin)
 - $r = c \cos n\theta$ for an integer n (rose curves with $2n$ petals if n is even and n petals if n is odd; c is the length of each petal)
 - $r = c\sqrt{\cos 2\theta}$ (lemniscate, i.e., infinity symbol; c is the length of the petal)
 - $r = a + b \cos \theta$ (limaçon, i.e., a slug; if $a = b$, a cardioid, i.e., a human heart)
 - $r = c\theta$ or $r = ce^\theta$ or $r = c\theta^2$ or any increasing function of θ (a spiral expanding around the origin)
 - $r = \frac{ed}{1+e \cos \theta}$ (a conic section with a focus at the origin and opening along the positive x -axis; an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$)

- Variations and techniques for getting the precise graphs:
 - If the equation only involves cosine, then replacing cosine with sine or replacing cosine with negative cosine results in the same shape rotated by some angle. Thus you only need to remember the basic forms for the polar curves.
 - The effect of changing $r = f(\theta)$ to $r = cf(\theta)$ is to expand or contract the entire graph by a factor of c .
 - To plot the standard curves, you only need to know a few special (r, θ) points. For a circle, lemniscate, limaçon, spiral, or conic section, you can get the entire graph just by knowing r at the cardinal angles $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. For a rose curve, it is sufficient just to know which θ gives the maxima, minima, and zeroes of r .
 - If you have a polar curve you don't recognize, you can figure out its graph by plotting (r, θ) pairs for those values of θ that maximize r , minimize r , or result in $r = 0$. This will always give the essential features of the graph.
 - Symmetry can help.
 - * If replacing (r, θ) with $(r, -\theta)$ gives the same equation, the graph is symmetric about the x -axis.
 - * If replacing (r, θ) with $(r, \pi - \theta)$ gives the same equation, the graph is symmetric about the y -axis.
 - * If replacing (r, θ) with either $(-r, \theta)$ or $(r, \theta + \pi)$ gives the same equation, the graph is symmetric about the origin.

- Slope of the tangent to a parametric equation:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

The cancelation of the dt terms makes this formula easy to remember.

- Concavity of a parametric equation:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

where $\frac{dy}{dx}$ is computed as above.

- Slope of the tangent to a polar equation $r(\theta)$: Using $x = r(\theta) \cos \theta$ and $y = r(\theta) \sin \theta$ and the product rule, you can derive

$$\frac{dy}{dx} = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r'(\theta) \cos \theta - r(\theta) \sin \theta}.$$

It's better to derive this formula as needed than to try to memorize it. *NEVER* make the mistake of saying the slope of a polar equation is $\frac{dr}{d\theta}$. A slope must always be expressed in rectangular coordinates.

- Arc length of a polar curve $r(\theta)$:

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \frac{dr^2}{d\theta}} d\theta.$$

- Area enclosed by a polar curve $r(\theta)$:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta.$$

Some subtleties:

- If $r \geq 0$ always, then you can find the area by integrating from 0 to 2π .
- If $r < 0$ for some values of θ , make sure you understand the graph before you try to integrate. Usually you can use symmetry (for example, when finding area in a rose, it's best to find the area in one petal, then multiply by the number of petals). Good bounds to use are consecutive values of θ which make $r = 0$; for example, to find the area in $\cos 3\theta$, integrate from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$ to get the area of one petal, then multiply by three to get the whole area.
- To find the area enclosed by two curves $r_1(\theta)$ and $r_2(\theta)$, first understand both graphs. (This is crucial! If you don't know what the region looks like, you'll probably get the wrong answer.) Find the angles of intersection by setting $r_1(\theta) = r_2(\theta)$, then compute $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r_2(\theta)^2 - \frac{1}{2} r_1(\theta)^2 d\theta$.
- Conic section formulas in rectangular coordinates (oriented along x -axis):
 - Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a > b$. Foci at $(c, 0)$ and $(-c, 0)$, where $c^2 = a^2 - b^2$. If the ellipse has $b > a$, then the foci are at $(0, c)$ and $(0, -c)$, where $c^2 = b^2 - a^2$.
 - Parabola: $4px = y^2$. Focus at $(p, 0)$, directrix at $x = -p$. If the parabola is of the form $4py = x^2$, then the focus is at $(0, p)$ and the directrix is $y = -p$.
 - Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Foci at $(c, 0)$ and $(-c, 0)$, where $c^2 = a^2 + b^2$. If the hyperbola is instead $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the foci are at $(0, c)$ and $(0, -c)$, where $c^2 = a^2 + b^2$ still. The asymptotes are $y = \pm \frac{b}{a}x$ in either case.
- Rotation of axes:

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta \end{aligned}$$

You can remember these formulas by making sure that they give the correct answer when $\theta = 0$ (then $x' = x$ and $y' = y$) and when $\theta = \frac{\pi}{2}$ (then $x' = y$ and $y' = -x$).

To get x and y in terms of x' and y' , just switch x with x' and y with y' in the above formulas, and change the sign of θ . This works because you rotate (x', y') axes by angle $-\theta$ to get back to (x, y) axes.

- If you have a quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

you can get rid of the cross term by rotating by angle θ , where θ is in the first quadrant and satisfies

$$\cot(2\theta) = \frac{A - C}{B}.$$

- After rotating axes, you will get a new equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0.$$

A shortcut for getting these coefficients:

$$A' = \frac{1}{2}((A + C) + (A - C) \cos 2\theta + B \sin 2\theta) = \frac{1}{2} \left(A + C + B \sqrt{1 + \left(\frac{A - C}{B} \right)^2} \right)$$

$$C' = \frac{1}{2}((A + C) + (C - A) \cos 2\theta - B \sin 2\theta) = \frac{1}{2} \left(A + C - B \sqrt{1 + \left(\frac{A - C}{B} \right)^2} \right)$$

$$D' = D \cos \theta + E \sin \theta$$

$$E' = -D \sin \theta + E \cos \theta$$

$$F' = F$$

The formulas for D' and E' can be remembered since they are the same as the transformation formulas for x' and y' . If D and E are zero, then you don't even need to know what θ is to transform the quadratic.

- The discriminant formula $B^2 - 4AC$ for a quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. If $B^2 - 4AC > 0$, the curve is a hyperbola. If $B^2 - 4AC = 0$, the curve is a parabola. If $B^2 - 4AC < 0$, the curve is an ellipse. Note the similarity to the quadratic formula. The discriminant is a very quick way to check the type of curve you're dealing with.
- Conic sections in polar coordinates:

$$r = \frac{ed}{1 + e \cos \theta}.$$

These curves have a focus at the origin and a directrix at $x = d$. If the directrix is horizontal instead of vertical, the $\cos \theta$ is replaced by $\sin \theta$. The *eccentricity* e is

$$e = \frac{\text{distance to focus}}{\text{distance to directrix}}$$

- The two fundamental numbers in this description are those obtained when $\cos \theta$ takes on its extreme values of 1 and -1 :

$$r_0 = \frac{ed}{e+1}, \quad r_1 = \frac{ed}{|1-e|}.$$

For an ellipse, r_0 is the closest approach to the focus while r_1 is the farthest point from the focus. For a hyperbola, r_1 is the vertex of the hyperbola branch opening away from the origin, while r_0 is the vertex of the hyperbola branch opening toward the origin. We always have $r_1 > r_0$. All other parameters of the conic section are determined in terms of r_0 and r_1 .

- If the major axis is the x -axis, then a , b , and c are determined by:

- * Ellipse: $a = \frac{r_0+r_1}{2}$, $c = \frac{r_1-r_0}{2}$, and $b = \sqrt{r_0r_1}$.

- * Hyperbola: $a = \frac{r_1-r_0}{2}$, $c = \frac{r_0+r_1}{2}$, and $b = \sqrt{r_0r_1}$.

You can always figure out these formulas if you know how to graph the ellipse, and you should be able to do this anyway.

- If we know a , b , and c , we can determine e and d as follows:

$$e = \frac{c}{a}, \quad d = \frac{a(1-e^2)}{e}$$

Definitions:

- *Ellipse.*

- Two-focus definition: The set of points such that the sum of distances to two fixed points (the *foci*) is constant.
- Focus-directrix definition: The set of points such that the quotient of the distance to a fixed point (*focus*) and the distance to a fixed line (*directrix*) is a constant less than one.

- *Parabola.* Focus-directrix definition: The set of points equidistant between a fixed point (*focus*) and a fixed line (*directrix*).

- *Hyperbola.*

- Two-focus definition: The set of points such that the difference of distances to two fixed points (the *foci*) is constant.
- Focus-directrix definition: The set of points such that the quotient of the distance to a fixed line (*directrix*) is a constant greater than one.

Be able to:

- Plot points in polar coordinates. Find the rectangular coordinates given polar coordinates. Find all possible polar coordinates corresponding to given rectangular coordinates. Transform polar equations to rectangular and vice versa.
- Sketch the graph of a curve given in polar coordinates by $r = f(\theta)$. Understand how to plot important points (the θ 's which maximize or minimize $f(\theta)$, as well as the zeroes of $f(\theta)$) and how to obtain the rest of the graph from these.
- Determine the x -axis, y -axis, or origin symmetry of a given polar curve $r = f(\theta)$.
- Find the tangent line to a parametric curve or polar curve at a given point. Determine whether a parametric curve is concave up or concave down by computing $\frac{d^2y}{dx^2}$.
- Find the arc length of a polar curve.
- Find the intersections of two polar curves. (Use the graph to help you with this, as they are not always obvious.)
- Find the area enclosed by a polar curve, or between two polar curves. (Use the graph for this as well, along with symmetries if any.)
- Identify the center, foci, vertices, and asymptotes (as applicable) for a conic section given in standard form. Be able to get the complete conic section knowing any two properties (for example, the location of the foci and the constant sum of the distances for an ellipse, or the asymptotes and the foci for a hyperbola).
- Identify the angle in the first quadrant by which to rotate a general conic section in order to obtain one in standard form. Identify the center of a translated conic section by completing the square. Describe the center, foci, vertices, and asymptotes (as applicable) in xy -coordinates once you've determined them in $x'y'$ -coordinates.
- Identify the vertices, center, foci, and asymptotes of an ellipse or hyperbola (as applicable) when given its focus, directrix, and eccentricity. (Use the fact that the vertices are given by r_0 and r_1 , the local extrema of the polar equation $r = \frac{ed}{1+e\cos\theta}$.) Identify the directrix and eccentricity of an ellipse or hyperbola given other information, such as the foci and vertices.

Review problems:

- 11.1: 61, 68. 11.2: 15, 16. 11.3: 21, 23.
11.4: 48–50. 11.5: 14, 31. 11.6: 15–17, 19–20.
- Chapter 11 Review: 1–2, 4–13, 15–18, 20–37, 39–41, 47–53.