Review Sheet for Third Exam

Mathematics 2300

November 15, 2006

The exam will cover all of Chapter 10.
No calculators of any kind will be allowed.

Definitions to know:

When we say know the definitions, this is a good indicator that you should *know the definitions*. Remember how you did with the definition of the limit on the last exam? Not so great, huh? Do better this time.

- Limit of a sequence: The sequence $\{a_n\}$ *converges to* L if, for any $\varepsilon > 0$, there is an integer N such that $|a_n - L| < \varepsilon$ for every $n \geq N$.

  Example: $\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1$ since for any $\varepsilon > 0$, we can choose N to be the first integer above $\frac{1}{\sqrt{\varepsilon}}$. Then if $n \geq N$, we will have $|a_n - L| = |\frac{1}{n^2}| < \varepsilon$.

- (Strictly) increasing: The sequence $\{a_n\}$ is *increasing* if $a_1 \leq a_2 \leq a_3 \leq \cdots$. It is *strictly increasing* if $a_1 < a_2 < a_3 < \cdots$. It is *eventually increasing* if deleting finitely many terms makes it an increasing sequence.

  Examples: $\{2n + (-1)^n\} = 1, 3, 3, 5, 5, 7, 7, \cdots$ is increasing but not strictly increasing.

  $\{n^2\} = 1, 4, 9, 16, \cdots$ is strictly increasing.

  $\{\frac{n!}{3^n}\} = \frac{1}{3}, \frac{2}{9}, \frac{6}{27}, \frac{24}{81}, \frac{120}{243}, \cdots$ is not strictly increasing, but it is eventually strictly increasing, since $\frac{6}{27}, \frac{24}{81}, \frac{120}{243}, \cdots$ is strictly increasing.

- (Strictly) decreasing: same as above, with all inequalities reversed.

- Bounded above: A sequence is *bounded above by* M if $a_n \leq M$ for every index n.

  Example: $\{\sin n\}$ is bounded above by $M = 1$.

- Bounded below: same as above, with inequality reversed.

- Partial sums: The *partial sums* of a series $\sum_{k=1}^{\infty} u_k$ are the terms of the sequence $s_n = u_1 + u_2 + \cdots + u_n$.

  Example: The partial sums of $\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+2}$ are $s_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$.

  Example: The partial sums of $\sum_{k=0}^{n} r^k$ are $s_n = \frac{1-r^{n+1}}{1-r}$.

- Convergence of a series: A series $\sum_{k=1}^{\infty} u_k$ *converges* if the sequence of partial sums has a limit. See definition of "limit" and of "partial sums." If the partial sums have no limit, then the series *diverges*.

  Example: The series $\sum_{k=0}^{\infty}(-1)^k$ diverges, since the partial sums are $\{s_n\} = 1, 0, 1, 0, 1, 0, \cdots$ which has no limit.

- Absolute convergence: The series $\sum_{k=1}^{\infty} u_k$ *converges absolutely* if the positive series $\sum_{k=1}^{\infty}|u_k|$ converges. Absolute convergence always implies (ordinary) convergence, but not the other way around.

  Examples: The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges, but it does not converge absolutely.

  The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely, since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ also converges.

- Taylor polynomials: The n^{th} *Taylor polynomial for $f(x)$ about x_o* is

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_o)}{k!}(x-x_o)^k = f(x_o)+f'(x_o)(x-x_o)+\frac{f''(x_o)}{2!}(x-x_o)^2+\cdots+\frac{f^{(n)}(x_o)}{n!}(x-x_o)^n.$$

  The n^{th} *Maclaurin polynomial for $f(x)$* is the n^{th} Taylor polynomial with $x_o = 0$.

  Example: The 3rd Taylor polynomial for $f(x) = \sqrt{x}$ about $x_o = 3$ is $p_3(x) = \sqrt{3} + \frac{1}{2\sqrt{3}}(x-3) - \frac{1}{2\cdot 4\cdot 3^{3/2}}(x-3)^2 + \frac{3}{6\cdot 8\cdot 3^{5/2}}(x-3)^3$.

- Taylor series: The *Taylor series for $f(x)$ about x_o* is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_o)}{k!}(x-x_o)^k.$$

  The *Maclaurin series for $f(x)$* is the Taylor series about $x_o = 0$.

  Example: The Taylor series for e^x about $x_o = 2$ is

$$\sum_{k=0}^{\infty} \frac{e^2(x-2)^k}{k!}$$

- Radius of convergence: The *radius of convergence* of a power series in $(x - x_o)$ is the number R such that the series converges if $|x - x_o| < R$ and diverges if $|x - x_o| > R$. It is usually found by using the ratio test on the power series. It may be 0 or ∞ or a finite positive number.

  Examples: The radius of convergence of $\sum_{k=0}^{\infty} k!x^k$ is $R = 0$. The radius of convergence of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is $R = \infty$. The radius of convergence of $\sum_{k=0}^{\infty} x^k$ is $R = 1$.

- Interval of convergence: The interval of convergence is the set of all x values such that the power series converges. It is always the interval between $x_o - R$ and $x_o + R$, possibly including the endpoints.

  Example: The interval of convergence of $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$ is $[0, 2)$.

Important theorems:

You will need to remember the statements of the theorems as well as how to apply them.

- Squeeze Theorem: If $a_n \le b_n \le c_n$ for every n, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then also the sequence in the middle converges to the same number.

  This theorem is good for sequences with complicated terms, like $b_n = 1 + \frac{1+(-1)^n}{n}$. We can use $a_n = 1$ and $c_n = 1 + 2/n$ and since both of these converge to 1, we must have $b_n \to 1$ as well.

- Monotone Convergence Theorem: Every bounded and eventually monotone sequence converges.

  This theorem is often used to determine convergence of recursive sequences, like $a_{n+1} = \sqrt{2 + a_n}$. We prove the sequence is monotone since $a_{n+2} - a_{n+1} = \sqrt{2 + a_{n+1}} - \sqrt{2 + a_n} = \frac{a_{n+1} - a_n}{\sqrt{2+a_{n+1}} + \sqrt{2+a_n}}$; thus the consecutive differences have the same sign. We prove the sequence is bounded above by 2 since if $a_n \le 2$, then $a_{n+1} = \sqrt{2 + a_n} \le \sqrt{2+2} = 2$ also. Thus we know the sequence converges.

- Divergence Theorem: If the series $\sum_{k=1}^{\infty} u_k$ converges, then the sequence of terms converges to zero: $\lim_{k\to\infty} u_k = 0$.

  This theorem is the basis of the k^{th}-term divergence test. For example, the series $\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$ diverges, since the terms converge to e. (See "Important sequences" below.)

- Alternating Series Convergence/Estimation Theorem: If the positive sequence $\{a_k\}$ is decreasing and convergent to zero, then the alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ is convergent. The error in the n^{th} partial sum is at most the magnitude of the $(n+1)^{\text{st}}$ term. In addition, the partial sums are too high if the last term is positive, and too low if the last term is negative.

  For example, the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$ is convergent since $\{\frac{1}{k^3}\}$ is decreasing and convergent to zero. If we approximate the sum with the first nine terms, then the error is at most $\frac{1}{10^3} = 0.001$, and the nine-term approximation is higher than the actual infinite sum, since the last term is $+1/9^3$.

- Taylor Remainder Estimation Theorem: The error in the n^{th} Taylor polynomial is

  $$|f(x) - p_n(x)| \le \frac{M_{n+1}}{(n+1)!} |x - x_o|^{n+1}$$

  where M_{n+1} is the maximum of $f^{(n+1)}$ on the interval from x_o to x.

  This formula can be remembered using the fact that it is essentially the absolute value of the next term in the Taylor polynomial (the difference being that instead of evaluating the $(n+1)^{\text{st}}$ derivative at x_o, we compute the maximum of it on an interval).

  The formula is useful for estimating errors in simple Taylor approximations. For example, suppose we want to approximate $\sqrt{4.4}$ with a quadratic Taylor polynomial. We expand around $x_o = 4$ with $f(x) = \sqrt{x}$ and obtain $p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ so that $p_2(4.4) = 2.0975$. The 3rd derivative of $f(x)$ is $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$, and the maximum on $[4, 4.4]$ occurs at $x = 4$ (since it's a decreasing function). So $M_3 = \frac{3}{2^8}$ and the error is at most $|\sqrt{4.4} - p_2(4.4)| \le M_3(0.4)^3/3! = \frac{1}{8000}$.

- Uniqueness of Taylor Series Theorem: If a function $f(x)$ is represented by a power series in $(x - x_o)$ about some interval $(x_o - R, x_o + R)$, then that series is the Taylor series for $f(x)$ about x_o.

  This theorem is important because it allows us to obtain Taylor (or Maclaurin) series without having to actually compute all derivatives. For example, we know $e^x = \sum_{k=0}^{\infty} x^k/k!$ for all x, and therefore $e^{-x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}/k!$ for all x. So this is the Maclaurin series for e^{-x^2}, and we would get the same thing if we actually computed all derivatives (although the current method is a lot quicker).

- Term-by-term Differentiation/Integration Theorem: If a function $f(x)$ has a Taylor series with a nonzero radius of convergence, then we can differentiate or integrate the Taylor series term by term inside the interval of convergence.

  This theorem is useful for more obtaining Taylor series for complicated functions from those of other functions. For example, to obtain a Maclaurin series for $\ln(1+x)$, we can start with the geometric series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ valid for $|x| < 1$, replace x with $-x$ to get $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$ valid for $|-x| < 1$, and then integrate term by term to get $\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k x^{k+1}/(k+1)$ also valid for $|x| < 1$. (In fact we gain convergence at the endpoint $x = 1$, but the radius of convergence does not change.)

Important sequence limits:

You don't need to memorize all of these, since you can derive them all by basic techniques. But be familiar with them.

- $\lim_{n\to\infty} r^n = \begin{cases} 0 & |r| < 1 \\ 1 & r = 1 \\ d.n.e. & \text{otherwise} \end{cases}$

- $\lim_{n\to\infty} n^p = \begin{cases} \infty & p > 0 \\ 0 & p < 0 \end{cases}$

- $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for any value of c.

- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. (This is proved, like all limits of the form 1^∞, by writing $y = e^{\ln y}$ and then using L'Hopital's Rule inside the exponential.)

- If $\{a_n\}$ is defined recursively by $a_{n+1} = f(a_n)$, and $\lim_{n\to\infty} a_n = L$ is known to exist, then $L = f(L)$.

- If $\lim_{x\to\infty} f(x) = L$, then also $\lim_{n\to\infty} f(n) = L$. This is not necessarily true the other way around.

Important series:

These series are important because other series are often compared to them to establish convergence or divergence. For example, the geometric series is the basis of the ratio and root tests, while the p-series is often used in the limit comparison test.

- Geometric series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ if $|r| < 1$; the series diverges if $|r| \geq 1$.
- p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$. If $p = 1$, this is the harmonic series.

Important Maclaurin series:

If you forget these, you can always derive them from the general formula, but it is probably easier just to remember them.

These series are important because most other functions with simple Taylor series (such as $\arctan x$, $\ln(1+x)$, $(1-x)^{-2}$, $\sinh x$, $\cosh x$, etc.) can be derived using these basic formulas, some algebra tricks, and the Uniqueness of Taylor Series theorem.

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$ (this is just the geometric series).
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all x.
- $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for all x.
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ for all x.

Convergence tests for series *with positive terms*:

You will need to know *which* test to use for a given series, as well as *how* to use it.

- k^{th} term test for divergence: If $\lim_{k\to\infty} u_k \neq 0$, then $\sum u_k$ diverges.

  When to use it: This is only sometimes useful. Generally you can tell from the form of the series whether the terms go to zero; if you're not immediately sure that $\lim_{k\to\infty} u_k = 0$, then check this first. For example, the series $\sum_{k=1}^{\infty} (1 - 2/k^2)^{k^2}$ diverges since the terms approach e^{-2}, and we might suspect this since the limit of the sequence looks like 1^∞.

  Also remember that it's quite possible for $u_k \to 0$ without the series converging (for example, the harmonic series), so this test does not work the other way around.

- Integral test: If $a_k = f(k)$, then the series $\sum_{k=1}^{\infty} a_k$ has the same behavior as the integral $\int_1^\infty f(x)\,dx$: either both converge or both diverge.

  When to use it: This test is best for series that look like substitution-integrals, like $\sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$, for which the integral $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ is easy to evaluate.

- Comparison test: Suppose $0 \leq a_k \leq b_k$ for every k. If $\sum b_k$ converges, then so does $\sum a_k$. If $\sum a_k$ diverges, then so does $\sum b_k$. There is no information given in any other case.

  When to use it: This test is a last resort, when the series is too complicated to deal with using any other test. It is good for dealing with complicated terms like $\sin^2 k$

or $2+(-1)^k$, since these are bounded above. For example, $\sum_{k=1}^{\infty} \sin^2 k/k^2$ converges by comparison with $\sum_{k=1}^{\infty} 1/k^2$, and $\sum_{k=1}^{\infty}(2+(-1)^k)/k$ diverges by comparison with $\sum_{k=1}^{\infty} 1/k$.

- Limit comparison test: If $a_k > 0$ and $b_k > 0$ for every k, and $\lim_{k\to\infty} \frac{a_k}{b_k}$ is a finite positive number, then the series $\sum a_k$ and $\sum b_k$ have the same behavior: either both converge or both diverge.

  When to use it: This is great for rational functions of k, such as $\sum_{k=1}^{\infty} \frac{3k^3+7k}{5k^6+2k^4+1}$. Use $b_k = 1/k^3$ to compare, and conclude that the given series also converges.

- Ratio test: Compute $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$. If $\rho < 1$, then $\sum a_k$ converges; if $\rho > 1$, then $\sum a_k$ diverges. If $\rho = 1$, use another test.

  When to use it: This is essential if the series involves a factorial, and is also good if the series has some number to a k^{th} power. For example, $\sum \frac{(2k)!}{k!(k+1)!}$ has $\rho = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{(k+1)(k+2)} = 4$, so the series diverges.

- Root test: Compute $\rho = \lim_{k\to\infty} |a_k|^{1/k}$. If $\rho < 1$, then $\sum a_k$ converges; if $\rho > 1$, then $\sum a_k$ diverges. If $\rho = 1$, use another test.

  When to use it: This is only really useful if a_k can be written as $(\text{stuff})^k$. For example, $\sum_{k=1}^{\infty} \left(\frac{2k+1}{3k+2}\right)^k$ has $\rho = \lim_{k\to\infty} \frac{2k+1}{3k+2} = \frac{2}{3}$, so the series converges.

Be able to:

- Prove that a given sequence is monotone. You need to do this if it's asked, and also if you want to prove an alternating series converges.

  If you have a formula $a_n = f(n)$ for a sequence, you can use one of three techniques:

  1. Compute $a_{n+1} - a_n$ and show that it's always positive/negative/nonpositive/nonnegative. (Good for algebraic formulas.)
  2. Compute a_{n+1}/a_n and show that it's always $> 1/ < 1/ \geq 1/ \leq 1$. (Good for sequences involving factorials or powers.)
  3. Compute $f'(x)$ and show that it's always positive/negative/nonpositive/nonnegative. (Great, as long as $f(x)$ actually is a differentiable function.)

  If you have a recursive sequence $a_{n+1} = F(a_n)$, you can sometimes try to prove $a_{n+2} - a_{n+1}$ always has the same sign as $a_{n+1} - a_n$ by writing the former in terms of the latter. This proves monotonicity.

- Compute the partial sums of a telescoping series explicitly, and use them to determine whether the telescoping series converges.

  Recall that a telescoping series is one for which the partial sums, written out, have all but a fixed number of terms canceling out. Common examples are $\sum \frac{1}{k(k+2)}$ (write it in partial fractions as $\sum \frac{1}{2k} - \frac{1}{2k+4})$ and $\sum \ln \frac{k+1}{k}$ (use the quotient rule for logs to write it as $\ln(k+1) - \ln k$).

- Prove that an alternating series converges by showing that the terms are decreasing (in absolute value) and converging to zero. Use this to estimate errors in approximating alternating series. See "Alternating Series Convergence/Estimation Theorem" above.

- Compute the n^{th} Taylor polynomial for a function around any base point x_o, and use the Taylor Remainder Estimation Theorem (see above) to determine the error when this is used to approximate the function at a fixed x.

- Find the radius of convergence R and interval of convergence for a given power series.

  The ratio test is always used to determine the radius of convergence. Once we have it, we just have to check whether the series converges when we plug in the endpoints $x_o - R$ and $x_o + R$ to determine whether the interval of convergence includes endpoints or not. (For this purpose, we always use one of the convergence tests other than the ratio or root tests.)

- Integrate a series term by term to approximate complicated integrals within the interval of convergence.

  For example,

$$\int_{-1}^{2} e^{-x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{-1}^{2} x^{2k}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2^{2k+1} - (-1)^{2k+1}}{2k+1}$$

  since the series converges for all x.

- Use known series to derive new series. For example, to derive a series for $\frac{1}{5+x}$ around $x_o = -3$, we can write

$$\frac{1}{5+x} = \frac{1}{2+(x+3)} = \frac{1}{2}\,\frac{1}{1-(-\frac{x+3}{2})} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)^k}{2^k}$$

  and this is valid if $|x+3| < 2$.

  As another example, to derive a series for xe^x around $x_o = 1$, we can write

$$xe^x = (x-1)e^x + e^x = e\cdot(x-1)e^{x-1} + e\cdot e^{x-1} = e\cdot\sum_{k=0}^{\infty} \frac{(x-1)^{k+1}}{k!} + e\cdot\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}$$

  and this is valid for all x.