

A GEOMETRIC RIGIDITY THEOREM FOR HYDRODYNAMICAL BLOWUP

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ABSTRACT. Suppose there is a smooth solution u of the 3D Euler equation with Lagrangian flow η such that for some Lagrangian path $\eta(t, x)$ and some time T , we have $\int_0^T |\omega(\eta(t, x))| dt = \infty$. Then in particular smoothness breaks down at time T by the Beale-Kato-Majda criterion. We know by the work of Arnold that the Lagrangian solution is a geodesic in the group of volume-preserving diffeomorphisms.

We show that either there is a sequence $t_n \nearrow T$ such that the corresponding geodesic fails to minimize length on each $[t_n, t_{n+1}]$, or there is a basis $\{e_1, e_2, e_3\}$ of $T_x M$ with e_3 parallel to the initial vorticity vector $\omega_0(x)$ such that the components of the stretching matrix $\Lambda(t, x) = (D\eta(t, x))^T D\eta(t, x)$ satisfy

$$\int_0^T \frac{\Lambda_{33}(\tau, x) d\tau}{\Lambda^{11}(\tau, x) + \Lambda^{22}(\tau, x)} < \infty$$

and

$$\lim_{t \rightarrow T} \frac{\int_0^t \Lambda^{11}(\tau, x) d\tau}{\int_0^t \Lambda^{22}(\tau, x) d\tau} = 0.$$

The former possibility can be studied in terms of the two-point minimization approach of Brenier on volume-preserving maps, while the latter gives a precise sense in which the vorticity vector tends to align with the intermediate eigenvector of the stretching matrix Λ .

Consider a three-dimensional ideal fluid flow on \mathbb{T}^3 described by a time-dependent velocity field $u: [0, T) \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$, satisfying the Euler equation

$$(1) \quad u_t + \nabla_u u = -\nabla p, \quad \operatorname{div} u \equiv 0, \quad u(0, x) = u_0(x).$$

Here the pressure is determined implicitly by $\Delta p = -\operatorname{div}(\nabla_u u)$. We assume u_0 is C^∞ .

The motion of particles is described by the flow map $\eta: [0, T) \times \mathbb{T}^3 \rightarrow \mathbb{T}^3$, defined by

$$\eta_t(t, x) = u(t, \eta(t, x)), \quad \eta(0, x) = x;$$

the incompressibility constraint $\operatorname{div} u = 0$ becomes

$$\det(D\eta(t, x)) \equiv 1 \text{ or } \eta^* \mu = \mu$$

in terms of the volume form μ . The particle map satisfies the Euler equation

$$(2) \quad \frac{\partial^2 \eta}{\partial t^2}(t, x) = -\nabla p(t, \eta(t, x))$$

with initial conditions $\eta(0, x) = x$ and $\eta_t(0, x) = u_0(x)$.

The vorticity vector field $\omega = \operatorname{curl} u$ satisfies

$$\omega_t + [u, \omega] = 0,$$

which leads to the important conservation law

$$(3) \quad \omega(t, \eta(t, x)) = D\eta(t, x)(\omega_0(x)).$$

We can think of η as a geodesic in the volumorphism group

$$\mathcal{D}_\mu(M) = \{\eta \in C^\infty(M, M) \mid \eta^* \mu = \mu\},$$

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as pointed out by Arnold [A], where the Riemannian metric is generated by the L^2 norm of the velocity field (i.e., the kinetic energy). If $s > \frac{5}{2}$, then the closure of $\mathcal{D}_\mu(M)$ in the Sobolev H^s topology is denoted by $\mathcal{D}_\mu^s(M)$, and it is a smooth submanifold of $H^s(M, M)$; furthermore the geodesic equation is a smooth ODE on this manifold, by results of Ebin-Marsden [EM]. Thus there is a smooth exponential map \exp_{id} from a neighborhood of 0 in the space of divergence-free H^s vector fields $T_{\text{id}}\mathcal{D}_\mu^s(M)$ to $\mathcal{D}_\mu^s(M)$ which takes u_0 to $\eta(1)$, with η satisfying (2).

Singular values of the exponential map are called *conjugate points*: those where the differential of the exponential map fails to be injective are *monoconjugate*, while those where the differential fails to be surjective are *epiconjugate* points. (In infinite dimensions, these are generally different; see Grossmann [Gr] and Biliotti et al. [BEPT] for an elaboration of these issues.) The derivative of the exponential map at 0 is the identity map, which implies by smoothness and the inverse function theorem that there is an H^s neighborhood of the 0 vector in which the exponential map is nonsingular. If η is a geodesic with $\eta(0) = \text{id}$, and $\eta(a)$ is conjugate to $\eta(0)$, then η cannot be minimizing on $[0, b]$ for any $b > a$. Intuitively, a conjugate point represents a family of geodesics all starting at the same point which meet to first order at the endpoint; the most familiar example is the north and south poles on a sphere. (For details, see a text such as Spivak [Sp], do Carmo [dC], or Lang [L].)

Along a geodesic η in $\mathcal{D}_\mu^s(\mathbb{T}^3)$, there are times $b > a > 0$ (either of which may be infinite) such that $\eta(t)$ is not conjugate to $\eta(0)$ for $t < a$, and $\eta(t)$ is epiconjugate to $\eta(0)$ for $a \leq t < b$. Furthermore for a countable dense set in $[a, b)$, $\eta(t)$ is monoconjugate to $\eta(0)$. See [P2] for an elaboration of this phenomenon. (In the cases where computations have been done explicitly, a is finite and b is infinite unless the initial velocity field u_0 is harmonic.)

Global existence of solutions to (1) is equivalent to geodesic completeness, which in turn is equivalent to the exponential map being defined on all of $T_{\text{id}}\mathcal{D}_\mu^s(M)$. The maximal time of existence T is finite if and only if the Beale-Kato-Majda criterion [BKM] is satisfied:

$$(4) \quad \int_0^T \sup_{x \in M} |\omega(t, x)| dt = \infty.$$

Several other criteria are known: see Constantin [Co] and Gibbon [Gi] for surveys of the literature, as well as Deng et al. [DHY] and Chae [Ch] for some recent results. However our analysis depends only on assuming a slightly strengthened version of (4).

Assumption 1. There is a point $x \in M$ such that

$$\int_0^T |\omega(t, x)| dt = \infty.$$

Clearly Assumption 1 is only possible if $\omega_0(x) \neq 0$, via (3). Hence we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ at $T_x M$ which is oriented in the usual way and such that e_3 is a positive multiple of $\omega_0(x)$. (This basis is of course unique only up to a rotation.) If we define the stretching matrix by

$$\Lambda(t, x) = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{pmatrix} \equiv D\eta(t, x)^T D\eta(t, x),$$

then $\Lambda_{33} = |D\eta(e_3)|^2$, so that Assumption 1 is obviously equivalent to

$$|\omega_0(x)| \int_0^T \sqrt{\Lambda_{33}(t, x)} dt = \infty.$$

As such, we can define a rescaled time variable by

$$(5) \quad s = |\omega_0(x)| \int_0^t \sqrt{\Lambda_{33}(\tau, x)} d\tau.$$

Obviously s goes from 0 to ∞ as t goes from 0 to T .

Now we define the property we want to establish for such a solution.

Definition 2. Suppose η is a geodesic in a Riemannian manifold, defined on a maximal interval $[0, T)$. We say that η has a *conjugate cascade* if there is a strictly increasing sequence of times $t_n \nearrow T$ such that $\eta(t_n)$ is conjugate to $\eta(t_{n+1})$.

By expanding the distances between t_n slightly (for example, considering $\tilde{t}_n = t_n + \frac{\varepsilon}{2^n}$ for some $\varepsilon > 0$), we can see that the conjugate cascade property implies that the geodesic fails to locally minimize length on ever-smaller intervals close to the blowup time T . (Note that since monoconjugate points are dense in epiconjugate points by [BEPT], we can equivalently assume $\eta(t_{n+1})$ and $\eta(t_n)$ are either monoconjugate or epiconjugate to each other.) In addition, since the minimum time between conjugate locations is governed by the maximum of the Riemannian sectional curvature, a conjugate cascade implies that the sectional curvature must approach positive infinity in directions containing the geodesic's tangent vector.

It should be noted that a conjugate cascade is very different from the pathological behavior of conjugate points discussed in [EMP] and elaborated in [P1] and [P2]. The difference here is that it is easy to have points conjugate to the *initial time*; it is much harder to find points conjugate to *each previous time*.

The main result of this paper is the following rigidity theorem, which says that among blowup scenarios, the conjugate cascade is “typical.” If blowup does not occur, then we get some fairly precise information about the structure of the stretching matrix Λ .

Theorem 3. *Suppose η is a solution of the Euler equation (2) with maximal time of existence $T < \infty$, satisfying Assumption 1 for some x . Then either η experiences a conjugate cascade, or there is a choice of orthonormal basis $\{e_1, e_2, e_3\}$ of $T_x M$ such that Λ satisfies the special conditions*

$$(6) \quad \lim_{t \rightarrow T} \frac{\int_0^t \Lambda^{11}(\tau) d\tau}{\int_0^t \Lambda^{22}(\tau) d\tau} = 0$$

and

$$(7) \quad \int_0^T \frac{\Lambda_{33}(\tau)}{\Lambda^{11}(\tau) + \Lambda^{22}(\tau)} d\tau < \infty.$$

(The two possibilities are not necessarily mutually exclusive.)

Roughly speaking (6) and (7) quantify the tendency of vorticity to align with the middle eigenvector of Λ , a well-known numerical observation [GGH].

We will prove this theorem through a sequence of lemmas in the remainder of this section. First we recall the following theorem proved in [P1].

Theorem 4. *Let η be a geodesic on $\mathcal{D}_\mu(M)$. If for some point x , the boundary value problem*

$$(8) \quad \frac{d}{dt} \left(\Lambda(t, x) \frac{dy}{dt} \right) + \omega_0(x) \times \frac{dy}{dt} = 0, \quad y(a) = 0, \quad y(b) = 0,$$

has a solution, then $\eta(a)$ and $\eta(b)$ are (epi)conjugate.

Again, by expanding the times slightly we can get the points to be monoconjugate.

Corollary 5. *The curve η experiences a conjugate cascade if there is a sequence $t_n \nearrow T$ such that for each n , equation (8) has a solution with $a = t_n$ and $b = t_{n+1}$.*

Hence the phenomenon is immediately reduced to the oscillation theory of the simple ODE (8). A system of the form (8) is said to be *oscillatory at $t = T$* if there is a sequence $t_n \nearrow T$ such that there are solutions for $a = t_n$ and $b = t_{n+1}$ for all n . (For this definition we may have $T = \infty$.)

In what remains we will analyze this ODE in great detail. Since from now on everything happens at the point x , we will omit reference to it to simplify notation.

First we reduce the three-dimensional system (8) to a two-dimensional system (in the plane orthogonal to the vorticity vector).

Lemma 6. *Suppose $\Lambda(t)$ is a positive-definite symmetric matrix with $\det \Lambda(t) \equiv 1$, and ω_0 is a positive number. Then the equation*

$$(9) \quad \frac{d}{dt} \left(\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) + \omega_0 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is oscillatory on $[0, T)$ if and only if the reduced equation

$$(10) \quad \frac{d}{dt} \left(\begin{bmatrix} \Lambda^{22}/\Lambda_{33} & -\Lambda^{12}/\Lambda_{33} \\ -\Lambda^{12}/\Lambda_{33} & \Lambda^{11}/\Lambda_{33} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) + \omega_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is oscillatory on $[0, T)$, where Λ^{ij} are the components of the inverse of Λ .

Proof. Let $0 \leq a < b < T$. By a well-known general principle (see e.g., Reid [Re]), a self-adjoint system of the form (9) or (10) has a solution with two zeroes in (a, b) if and only if there is a y with $y(a) = y(b) = 0$ such that the corresponding index form $I(y, y)$ is negative.

For (9), the index form is

$$(11) \quad I_3(y, y) = \int_a^b (\Lambda_{11}\dot{y}_1^2 + \Lambda_{22}\dot{y}_2^2 + \Lambda_{33}\dot{y}_3^2 + 2\Lambda_{12}\dot{y}_1\dot{y}_2 + 2\Lambda_{13}\dot{y}_1\dot{y}_3 + 2\Lambda_{23}\dot{y}_2\dot{y}_3 + \omega_0(y_1\dot{y}_2 - y_2\dot{y}_1)) dt,$$

while for (10) the index form is

$$(12) \quad I_2(y, y) = \int_a^b \left(\frac{1}{\Lambda_{33}} (\Lambda^{22}\dot{y}_1^2 - 2\Lambda^{12}\dot{y}_1\dot{y}_2 + \Lambda^{11}\dot{y}_2^2) + \omega_0(y_1\dot{y}_2 - y_2\dot{y}_1) \right) dt.$$

Completing the square in (11), we get

$$I_3(y, y) = I_2(y, y) + \int_a^b \Lambda_{33} \left[\dot{y}_3 + \frac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \frac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2 \right]^2 dt.$$

The last term is clearly nonnegative, so that $I_2(y, y) \leq I_3(y, y)$; hence (10) is oscillatory if (9) is.

To prove the other direction, observe by Cauchy-Schwarz that

$$\int_a^b \Lambda_{33} \left[\dot{y}_3 + \frac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \frac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2 \right]^2 dt \geq \left(\int_a^b \left(\frac{\Lambda_{13}}{\Lambda_{33}}\dot{y}_1 + \frac{\Lambda_{23}}{\Lambda_{33}}\dot{y}_2 \right) dt \right)^2 / \int_a^b \frac{dt}{\Lambda_{33}},$$

using the fact that $y_3(a) = y_3(b) = 0$.

Now if a solution of (10) has infinitely many zeroes on $[0, T)$, we can given any a find a $b \in (a, T)$ such that some solution \bar{y} has three zeroes at $t = a$, $t = b$, and $t = c \in (a, b)$. Writing (10) as $Ly = 0$, we see that 0 is an eigenvalue of the self-adjoint operator L on $C_0[a, b]$. Since \bar{y} is an eigenfunction which vanishes in (a, b) , we know 0 is not the smallest eigenvalue of $-L$. (See for example Courant-Hilbert [CH].) So let $\tilde{\lambda} < 0$ be an eigenvalue with eigenfunction \tilde{y} , so that $L\tilde{y} = -\tilde{\lambda}\tilde{y}$.

Choose $c, d \in \mathbb{R}$ with $c^2 + d^2 \neq 0$ such that

$$c \int_a^b \left(\dot{\bar{y}}_1 \frac{\Lambda_{13}}{\Lambda_{33}} + \dot{\bar{y}}_2 \frac{\Lambda_{23}}{\Lambda_{33}} \right) dt + d \int_a^b \left(\dot{\tilde{y}}_1 \frac{\Lambda_{13}}{\Lambda_{33}} + \dot{\tilde{y}}_2 \frac{\Lambda_{23}}{\Lambda_{33}} \right) dt = 0.$$

Set $y = c\bar{y} + d\tilde{y}$; then we have

$$I_3(y, y) = I_2(y, y) = - \int_a^b \langle y, Ly \rangle dt = \int_a^b \langle c\bar{y} + d\tilde{y}, d\tilde{\lambda}\tilde{y} \rangle dt = d^2\tilde{\lambda} \int_a^b |\tilde{y}|^2 dt \leq 0.$$

Therefore equation (9) has a solution with at least two zeroes in $[a, b]$. \square

Now working with (10), define a new time variable s as in (5). Then the index form (12) takes the form

$$(13) \quad I_2(y, y) = \int_{\bar{a}}^{\bar{b}} \left(\Phi_{11} \left(\frac{dy_1}{ds} \right)^2 + 2\Phi_{12} \frac{dy_1}{ds} \frac{dy_2}{ds} + \Phi_{22} \left(\frac{dy_2}{ds} \right)^2 + y_1 \frac{dy_2}{ds} - y_2 \frac{dy_1}{ds} \right) ds,$$

where

$$(14) \quad \Phi_{11} = \frac{\Lambda^{22}}{\sqrt{\Lambda_{33}}}, \quad \Phi_{12} = -\frac{\Lambda^{12}}{\sqrt{\Lambda_{33}}}, \quad \text{and} \quad \Phi_{22} = \frac{\Lambda^{11}}{\sqrt{\Lambda_{33}}}.$$

Observe that since $\det \Lambda = 1$, we have $\Phi_{11}\Phi_{22} - \Phi_{12}^2 \equiv 1$.

Now set $\Phi(t) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12} & \Phi_{22} \end{bmatrix}$ and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Obviously the index form (13) is zero on $[\bar{a}, \bar{b}]$ for some y if and only if the solution of the 2×2 matrix equation

$$(15) \quad \frac{d}{ds} \left(\Phi \frac{dY}{ds} \right) + J \frac{dY}{ds} = 0, \quad Y(\bar{a}) = 0, \quad Y'(\bar{a}) = \text{id}$$

has a solution with $\det(Y(\bar{b})) = 0$, since we can then find a vector z_0 with $Y(\bar{b})(z_0) = 0$; in that case $y(t) = Y(t)(z_0)$ is the desired solution of (10).

This leads to a useful criterion in terms of a first-order 2×2 ODE system.

Lemma 7. *Equation (15) is oscillatory at $s = \infty$ if and only if there is a sequence $s_n \nearrow \infty$ such that the solution of the matrix equation*

$$(16) \quad \frac{dW}{ds} + J\Phi(s)^{-1}W(s) = 0$$

with $W(0) = \text{id}$ satisfies

$$(17) \quad \text{Tr}(W(s_{n+1})W(s_n)^{-1}) = 2$$

for every n .

Proof. We just define $W(s) = \Phi(s)Y'(s)$. Clearly (15) reduces to (16) under this substitution, with $W(s_n) = \Phi(s_n)$. On the other hand if we replace only the term in parentheses from (15) with W , we obtain

$$\frac{dW}{ds} + J \frac{dY}{ds} = 0,$$

from which we obtain

$$W(s_{n+1}) + JY(s_{n+1}) = W(s_n)$$

since $Y(s_n) = 0$. Since

$$\det(-JY(s_{n+1})) = -\det(Y(s_{n+1})) = 0,$$

we see that $\det(W(s_{n+1}) - W(s_n)) = 0$.

Now the general formula

$$\frac{d}{ds} \ln(\det W) = \text{Tr} \left(\frac{dW}{ds} W^{-1} \right)$$

implies that $\det W$ is constant, and since $W(s_n) = \Phi(s_n)$ we see that $\det W(s) \equiv \det \Phi(s_n) = 1$. So our requirement is that $\det(W(s_{n+1})W(s_n)^{-1} - \text{id}) = 0$.

Now for the matrix $A = W(s_{n+1})W(s_n)^{-1}$, we know $\det A = 1$, so that if $\det(A - \text{id}) = 0$, the matrix A must have eigenvalue 1 with multiplicity two, which implies $\text{Tr} A = 2$ by the Cayley-Hamilton theorem. This argument can obviously be reversed, so we get a necessary and sufficient condition (17). \square

Now the idea of what we want to do next is the following. Express $\Phi(s)$ in terms of its eigenvalues e^λ and $e^{-\lambda}$ and eigenbasis $\begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix}$ as

$$\Phi = \begin{bmatrix} \cosh \lambda + \sinh \lambda \cos 2\gamma & \sinh \lambda \sin 2\gamma \\ \sinh \lambda \sin 2\gamma & \cosh \lambda - \sinh \lambda \cos 2\gamma \end{bmatrix}$$

and express $W(s)$ as

$$W(s) = \cosh \psi \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} + \sinh \psi \begin{bmatrix} \cos \beta & -\sin \beta \\ -\sin \beta & -\cos \beta \end{bmatrix}.$$

Then equation (16) takes the form

$$(18) \quad \frac{d\alpha}{ds} = \cosh \lambda - \sinh \lambda \tanh \psi \cos(\alpha + \beta + 2\gamma),$$

$$(19) \quad \frac{d\beta}{ds} = \cosh \lambda - \sinh \lambda \coth \psi \cos(\alpha + \beta + 2\gamma),$$

$$(20) \quad \frac{d\psi}{ds} = -\sinh \lambda \sin(\alpha + \beta + 2\gamma).$$

We observe that α is strictly increasing, and since the trace condition (17) translates into

$$(21) \quad \cosh \psi(s_{n+1}) \cosh \psi(s_n) \cos[\alpha(s_{n+1}) - \alpha(s_n)] \\ - \sinh \psi(s_{n+1}) \sinh \psi(s_n) \cos[\beta(s_{n+1}) - \beta(s_n)] = 1,$$

it is easy to see that $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ is sufficient for a conjugate cascade. Hence if a cascade doesn't happen, we must have

$$\int_0^\infty [\cosh \lambda - \sinh \lambda \tanh \psi \cos(\alpha + \beta + 2\gamma)] ds < \infty.$$

Roughly speaking, this should imply that $|\lambda| \rightarrow \infty$, $|\psi| \rightarrow \infty$, and $\alpha + \beta + 2\gamma \rightarrow n\pi$ for some integer n . Thus both α and β roughly converge, and thus so does γ .

These statements are not quite correct since $\int_0^\infty |f(s)| ds < \infty$ does not imply $f(s) \rightarrow 0$ as $s \rightarrow \infty$, but we can still get convergence in an average sense. The other problem is that the variables γ and β are not well-defined when Φ or W happens to be the identity, while λ and ψ are not smooth at such times. Hence we will work directly with the components of Φ and W , although we always have the trigonometric coordinates in mind.

With that said, we now write

$$(22) \quad \Phi = \begin{bmatrix} p+r & q \\ q & p-r \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} a+c & b+d \\ d-b & a-c \end{bmatrix},$$

where a, b, c, d, p, q, r are all smooth functions of the rescaled time parameter s . The conditions $\det \Phi \equiv 1$ and $\det W \equiv 1$ imply

$$(23) \quad p^2 = 1 + q^2 + r^2 \quad \text{and} \quad a^2 + b^2 = 1 + c^2 + d^2.$$

Theorem 8. *Suppose η is a geodesic in $\mathcal{D}_\mu(M)$ satisfying Assumption 1 which does not experience a conjugate cascade as $t \nearrow T$.*

Then there is a basis $\{e_1, e_2, e_3\}$ of $T_x M$ such that, in terms of the new time variable s defined by (5), and the variables p, q, r defining Φ in (22), we have

$$(24) \quad \int_0^\infty p - \sqrt{p^2 - 1} ds < \infty$$

as well as

$$(25) \quad \lim_{\Sigma \rightarrow \infty} \frac{\int_0^\Sigma [\sqrt{q^2 + r^2} - r] ds}{\int_0^\Sigma [\sqrt{q^2 + r^2}] ds} = 0.$$

Proof. Set

$$(26) \quad f = 2(ad - bc), \quad g = 2(ac + bd), \quad h = a^2 + b^2 + c^2 + d^2,$$

where a, b, c, d are the components of W as in (22). Then equation (23) implies

$$h^2 = 1 + f^2 + g^2.$$

We compute that

$$(27) \quad \begin{aligned} \frac{da}{ds} &= -pb - qc + rd, & \frac{db}{ds} &= pa - rc - qd, \\ \frac{dc}{ds} &= -qa - rb + pd, & \frac{dd}{ds} &= ra - qb - pc. \end{aligned}$$

Now equation (23) implies that $a^2 + b^2 \neq 0$ for all time, and hence the functions $\rho = \sqrt{a^2 + b^2}$ and α defined by $\rho \cos \alpha = a$ and $\rho \sin \alpha = b$ are well-defined and smooth for all time. (We set $\alpha(0) = 0$ to normalize, since $a(0) = 1$ and $b(0) = 0$.)

Equations (27) imply that

$$\frac{d\alpha}{ds} = \frac{a\dot{b} - b\dot{a}}{a^2 + b^2} = p - \frac{qf + rg}{2(a^2 + b^2)},$$

using (26). Now $2(a^2 + b^2) = h + 1$, so we get

$$(28) \quad \begin{aligned} \frac{d\alpha}{ds} &= p - \frac{qf + rg}{h + 1} = p - \sqrt{p^2 - 1} + \sqrt{q^2 + r^2} - \frac{qf + rg}{h + 1} \\ &= \left(p - \sqrt{p^2 - 1}\right) + \sqrt{q^2 + r^2} \left(1 - \sqrt{\frac{h-1}{h+1}}\right) + \frac{\sqrt{(q^2 + r^2)(f^2 + g^2)} - (qf + rg)}{h + 1}. \end{aligned}$$

Clearly each term of (28) is nonnegative, with the first and second being strictly positive. So α is strictly increasing. We next want to show that if α approaches infinity, then there is a conjugate cascade.

We compute that the trace condition (17) is equivalent to

$$a(s_n)a(s_{n+1}) + b(s_n)b(s_{n+1}) - c(s_n)c(s_{n+1}) - d(s_n)d(s_{n+1}) = 1,$$

which in terms of ρ and α is equivalent to $F(s_{n+1}) = 1$ where

$$F(s) = \rho(s)\rho(s_n) \cos(\alpha(s) - \alpha(s_n)) - \langle \phi(s_n), \phi(s) \rangle,$$

with $\phi(s) = (c(s), d(s))$. Observe that

$$|\phi(s)| = \sqrt{c(s)^2 + d(s)^2} = \sqrt{\rho(s)^2 - 1}.$$

Now if $\lim_{s \rightarrow \infty} \alpha(s) = +\infty$, then we can choose $s' > s_n$ such that $\alpha(s') = \alpha(s_n) + 2\pi$, and get that $F(s') \geq \rho(s)\rho(s_n) - \sqrt{(\rho(s)^2 - 1)(\rho(s_n)^2 - 1)} \geq 1$. We can also choose $s'' > s_n$ such that $\alpha(s'') = \alpha(s_n) + \pi$, so that $F(s'') \leq -\rho(s)\rho(s_n) + \sqrt{(\rho(s)^2 - 1)(\rho(s_n)^2 - 1)} \leq -1$. So between s'' and s' there is a solution s_{n+1} of $F(s_{n+1}) = 1$.

Note that the inequalities on F follow from $\sqrt{(1+x^2)(1+y^2)} - xy \geq 1$, which follows from the Cauchy-Schwarz inequality in the form $1 + 2xy + x^2y^2 \leq 1 + x^2 + y^2 + x^2y^2$.

Hence if η does not experience a conjugate cascade, then $\lim_{s \rightarrow \infty} \alpha(s) < \infty$, which implies that each of the three nonnegative terms in (28) has a finite integral. Thus we get (24), in addition to

$$(29) \quad \int_0^\infty \sqrt{q^2 + r^2} \left(1 - \sqrt{\frac{h-1}{h+1}}\right) ds < \infty$$

and

$$(30) \quad \int_0^\infty \frac{1}{h+1} \left(\sqrt{(q^2 + r^2)(f^2 + g^2)} - (qf + rg)\right) ds < \infty.$$

We first explore the consequences of (29). Define $\lambda(s) \geq 0$ by $\sqrt{p^2 - 1} = \sinh \lambda$. Then λ is certainly continuous, and we have $p = \cosh \lambda$, so that (24) says that

$$(31) \quad \int_0^\infty e^{-\lambda(s)} ds < \infty.$$

Hence (29) says that

$$\int_0^\infty (e^\lambda - e^{-\lambda})u ds < \infty,$$

where $u = 1 - \sqrt{\frac{h-1}{h+1}}$. Now since $0 < u \leq 1$, we already know that $\int_0^\infty e^{-\lambda}u ds < \infty$, so (29) really tells us that $\int_0^\infty e^\lambda u ds < \infty$. Now (31) implies that $\int_0^\infty e^{\lambda(s)} ds = \infty$, so if we define a new rescaled time variable by $d\sigma = e^{\lambda(s)} ds$, then $\sigma \rightarrow \infty$ as $s \rightarrow \infty$. So (29) can be written as

$$(32) \quad \int_0^\infty u(\sigma) d\sigma < \infty.$$

We can compute using (27) that $\frac{dh}{ds} = 2(rf - qg)$, so that using $h^2 - 1 = f^2 + g^2$ we get

$$(33) \quad \left| \frac{du}{d\sigma} \right| = |2u - u^2| \frac{|rf - qg|}{\sqrt{(f^2 + g^2)(q^2 + r^2)}} \leq 1.$$

It is easy to see that (32) and (33) together imply that $\lim_{\sigma \rightarrow \infty} u(\sigma) \searrow 0$, and hence that $\lim_{s \rightarrow \infty} u(s) \searrow 0$. Since $h(s) = \frac{1}{u(s)} + \frac{1}{2-u(s)} - 1$, we see that $\lim_{s \rightarrow \infty} h(s) = +\infty$.

Finally we look at the consequence of (30). Since $\lim_{s \rightarrow \infty} h(s) = \infty$, there is some M such that $h(s) > 1$ for $s > M$; in particular $f^2 + g^2 > 0$, so that it makes sense to define ξ for $s > M$ by $\frac{f}{\sqrt{f^2 + g^2}} = \cos \xi$ and $\frac{g}{\sqrt{f^2 + g^2}} = \sin \xi$, and this ξ is smooth on (M, ∞) . It is easy to compute that

$$\frac{d\xi}{ds} = 2p - \frac{2h(qf + rg)}{h^2 - 1} = 2p - \frac{2h}{\sqrt{h^2 - 1}}(q\bar{f} + r\bar{g}),$$

where $\bar{f} = \frac{f}{\sqrt{f^2 + g^2}}$ and $\bar{g} = \frac{g}{\sqrt{f^2 + g^2}}$. Now we have

$$\frac{d\xi}{ds} \leq 2[p - \sqrt{p^2 - 1}] + 2 \left[\left(\frac{h}{\sqrt{h^2 - 1}} - 1 \right) (q\bar{f} + r\bar{g}) \right] + \left[\sqrt{q^2 + r^2} - (q\bar{f} + r\bar{g}) \right],$$

and we know that each of the terms is (absolutely) integrable. Thus $\lim_{s \rightarrow \infty} \xi(s)$ exists. In particular $\lim_{s \rightarrow \infty} \bar{f}(s)$ and $\lim_{s \rightarrow \infty} \bar{g}(s)$ exist. Now we never specified the specific directions of $\{e_1, e_2\}$ (the basis of the orthogonal complement of $\omega_0(x)$), so we might as well now rotate them so that $\lim_{s \rightarrow \infty} \bar{f}(s) = 0$ and $\lim_{s \rightarrow \infty} \bar{g}(s) = 1$.

Now from (30), using (29), we can easily show that

$$(34) \quad \int_M^\infty [\sqrt{q^2 + r^2} - (q\bar{f} + r\bar{g})] ds < \infty.$$

Now write for any $\Sigma > M$

$$\frac{\int_M^\Sigma [\sqrt{q^2 + r^2} - (q\bar{f} + r\bar{g})] ds}{\int_M^\Sigma \sqrt{q^2 + r^2} ds} + \frac{\int_M^\Sigma [q\bar{f} + r(\bar{g} - 1)] ds}{\int_M^\Sigma \sqrt{q^2 + r^2} ds} = \frac{\int_M^\Sigma [\sqrt{q^2 + r^2} - r] ds}{\int_M^\Sigma \sqrt{q^2 + r^2} ds}.$$

Since $\sqrt{q^2 + r^2} = \sinh \lambda$ and $\int_0^\infty e^\lambda ds = \infty$, we know the denominator approaches infinity as $\Sigma \rightarrow \infty$. The second quotient approaches zero as $\Sigma \rightarrow \infty$ since $\bar{f} \rightarrow 0$ and $\bar{g} \rightarrow 1$, while the first quotient approaches zero because of (34). We therefore conclude that

$$\lim_{\Sigma \rightarrow \infty} \frac{\int_M^\Sigma [\sqrt{q^2 + r^2} - r] ds}{\int_M^\Sigma \sqrt{q^2 + r^2} ds} = 0.$$

Because of this, and because $\lim_{\Sigma \rightarrow \infty} \int_0^\Sigma \sqrt{q^2 + r^2} ds = \infty$, we also have (25). \square

Finally we translate the results of Theorem 8 into the original time variable and in terms of more easily measured quantities.

Corollary 9. *Suppose η is a geodesic in $\mathcal{D}_\mu(M)$ satisfying Assumption 1. If η does not experience a conjugate cascade, then there is an oriented orthonormal basis $\{e_1, e_2, e_3\}$ of $T_x M$ such that $\omega_0(x)$ is parallel to e_3 , and the components Λ_{ij} of $\Lambda = (D\eta)^T(D\eta)$ and the components Λ^{ij} of its inverse satisfy the following:*

$$(35) \quad \int_0^T \frac{\Lambda_{33}(\tau)}{\Lambda^{11}(\tau) + \Lambda^{22}(\tau)} d\tau < \infty$$

and

$$(36) \quad \lim_{t \rightarrow T} \frac{\int_0^t \Lambda^{11}(\tau) d\tau}{\int_0^t \Lambda^{22}(\tau) d\tau} = 0.$$

Proof. By (14) and (22), we have

$$p = \frac{\Lambda^{11} + \Lambda^{22}}{2\sqrt{\Lambda_{33}}},$$

and using $ds = \omega_0 \sqrt{\Lambda_{33}} dt$, the inequality (24) becomes

$$\int_0^T \left(\Lambda^{11} + \Lambda^{22} - \sqrt{(\Lambda^{11} + \Lambda^{22})^2 - 4\Lambda_{33}} \right) dt < \infty.$$

Rationalizing, we get

$$\int_0^T \frac{\Lambda_{33}}{\Lambda^{11} + \Lambda^{22} + \sqrt{(\Lambda^{11} + \Lambda^{22})^2 - 4\Lambda_{33}}} dt < \infty,$$

so that in particular the integral (35) is finite.

We can replace the term $\sqrt{q^2 + r^2}$ appearing in (25) with p , since the integral of $p - \sqrt{q^2 + r^2}$ is finite by (24). So (25) becomes

$$\lim_{\Sigma \rightarrow \infty} \frac{\int_0^\Sigma (p - r) ds}{\int_0^\Sigma (p - r) + (p + r) ds} = 0.$$

The only way this can happen is if

$$\lim_{\Sigma \rightarrow \infty} \frac{\int_0^\Sigma (p - r) ds}{\int_0^\Sigma (p + r) ds} = 0,$$

which after changing the time variable is equivalent to (36). \square

Notice that a conjugate cascade is closely related to the uniqueness problem for minimizers in the space of volume-preserving maps (VPM). The general minimization problem on VPM was proposed by Brenier [B1] and has been studied further by him [B2], Shnirelman [Sh2], and Ambrosio-Figalli [AF]. The space VPM is the closure in the L^2 topology of the volumorphism group, and is a metric space in this topology. If we think of VPM as a closed subset of the space of all L^2 maps from M^3 to itself, we can find a Hölder-type bound on the intrinsic distance in VPM in terms of the extrinsic L^2 distance by $d_{\text{VPM}}(\eta, \xi) \leq C(d_{L^2}(\eta, \xi))^\alpha$ for some $\alpha < 1$. (These results were proved by Shnirelman [Sh1].) A local uniqueness result for minimizing paths in the L^2 topology of VPM would immediately preclude a conjugate cascade, since the intrinsic L^2 distance between successive conjugate point locations goes to zero by the above Hölder estimate, while the geodesic is non-minimizing on any interval $[b, T]$ for $b < T$. As of this writing, such a local uniqueness result is unknown.

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