Consider an ideal fluid filling up a compact portion M of $\mathbb{R}^n$ or the torus $\mathbb{T}^n$. If M has boundary ∂M, we will call $\hat{n}$ the outward normal vector field.

With no viscosity or compressibility, the velocity field X of the fluid will satisfy the Euler equation

$$\frac{\partial X}{\partial t} + X \cdot \nabla X = -\nabla p, \qquad \operatorname{div} X = 0,\ X \cdot \hat{n} = 0$$

The pressure function p is determined implicitly. Taking divergence of both sides...

$$\Delta p = -\operatorname{div}(X \cdot \nabla X), \qquad \nabla p \cdot \hat{n} = -(X \cdot \nabla X) \cdot \hat{n}$$

This Neumann problem can always be solved for p.

The motion of the fluid particles can then be found from the flow equation:

$$\frac{\partial \eta}{\partial t}(t,x) = X\big(t, \eta(t,x)\big), \qquad \eta(0,x) = x$$

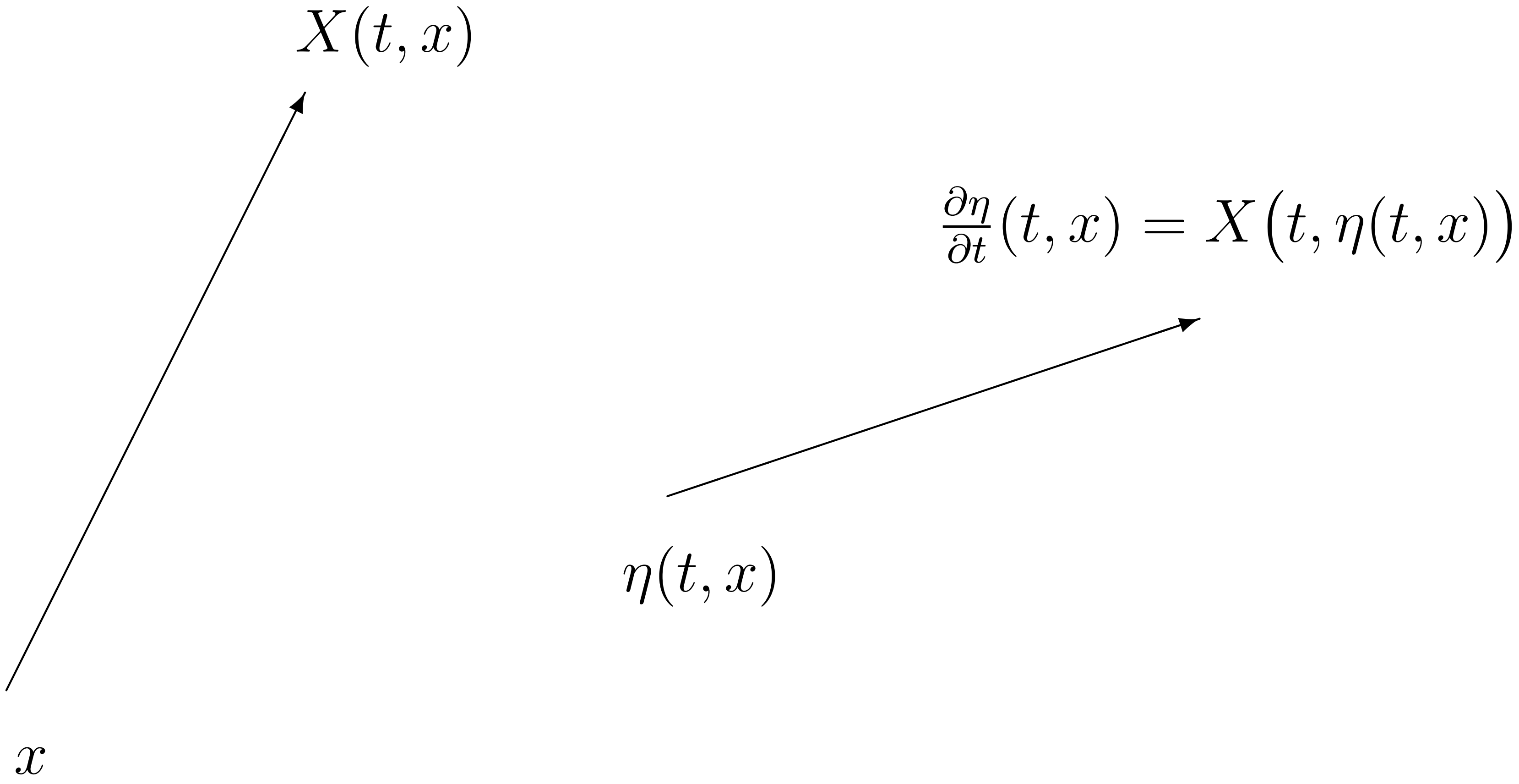
Since $\operatorname{div} X = 0$, we know η is volume-preserving:

$$J(\eta) \equiv 1$$

Thus as long as X exists, η will be a diffeomorphism of M. More specifically, it will be a *volumorphism*.

So we can think of $\eta(t)$ as being a path in the space of volumorphisms.

Notice that $\frac{\partial \eta}{\partial t}$ is not a vector field, but $\frac{\partial \eta}{\partial t} \circ \eta^{-1}$ is.



$X(t,x)$

$\frac{\partial \eta}{\partial t}(t,x) = X\big(t, \eta(t,x)\big)$

$\eta(t,x)$

x

In terms of the particle paths, the Euler equation can be written as

$$\frac{\partial^2}{\partial t^2} = -\nabla p \circ \eta$$

We will derive the Euler equation from a variational principle.

Define the L^2 distance in the space of volumorphisms by

$$D(\eta, \xi)^2 = \int_M |\eta(x) - \xi(x)|^2 \, dx.$$

Then the length of the curve $\eta(t)$ for $0 \leq t \leq T$ is

$$L = \int_0^T \sqrt{\int_M \left| \frac{\partial \eta}{\partial t} \right|^2 \, dx} \, dt.$$

If we have a family of paths $\eta(t,\varepsilon)$ of volumorphisms with $\eta(t,0)$ the desired minimizer, then we want $L'(0)=0$. We have

$$L'(\varepsilon) = 2\int_0^T \frac{\int_M \eta_{t\varepsilon}\cdot\eta_t\,dx}{\sqrt{\int_M |\eta_t|^2\,dx}}\,dt$$

so that we need

$$0 = -\int_0^T \frac{\int_M \eta_{tt}\cdot\eta_\varepsilon|_{\varepsilon=0}\,dx}{\sqrt{\int_M |\eta_t|^2\,dx}}\,dt$$

If we assume that the curve $t \mapsto \eta(t)$ is parametrized proportional to arc length, then

$$\int_M |\eta_t|^2 \, dx = k$$

for some constant k independent of t, and then the extremal condition $L'(0) = 0$ is equivalent to

$$\int_0^T \int_M \eta_{tt} \cdot \eta_\varepsilon \, dx \, dt = 0$$

Now recall that η_{tt} and η_ε are not vector fields. However $\eta_{tt} \circ \eta^{-1}$ and $\eta_\varepsilon \circ \eta^{-1}$ are. We want the integral in terms of vector fields, so we can compute things more easily.

Let's call $Y(t) = \eta_\varepsilon\big|_{\varepsilon=0} \circ \eta^{-1}$. Then since each $\eta(t,\varepsilon)$ is a volumorphism, we must have $\operatorname{div} Y = 0$ for every t.

We must have

$$\int_0^T \int_M \Big(\eta_{tt} \circ \eta^{-1} \cdot Y(t)\Big) \circ \eta \, dx \, dt = 0$$

Since η is a volumorphism, we know $J(\eta) \equiv 1$, so by the change-of-variables formula, we have

$$\int_0^T \int_M \left(\eta_{tt} \circ \eta^{-1}\right) \cdot Y(t) \, dx \, dt = 0$$

Now $Y(t)$ can be any divergence-free vector field, and thus our condition is

$\eta_{tt} \circ \eta^{-1}$ is L^2-orthogonal to every divergence-free vector field.

This is a good time to discuss the Weyl decomposition.

Suppose V is any vector field on M. (Not necessarily divergence-free or tangent to the boundary.)

We want to write

$$V = U + \nabla f$$

where U is divergence-free and tangent to the boundary.

To do this, solve the Neumann problem

$$\Delta f = \operatorname{div} V, \qquad \nabla f \cdot \hat{n} = V \cdot \hat{n}$$

The solution f is unique up to a constant, so ∇f is unique. Define $U = V - \nabla f$, and we're done.

We will call $U = P(V)$. This is the Leray projection.

Notice that every divergence-free vector field, tangent to the boundary, is L^2-orthogonal to every gradient. This comes from

$$\begin{aligned}\int_M U \cdot \nabla f \, dx &= \int_M \operatorname{div}(fU) \, dx \\ &\qquad - \int_M f \operatorname{div} U \, dx && \text{(product rule)} \\ &= \int_M \operatorname{div}(fU) && \text{(since } \operatorname{div} U = 0\text{)} \\ &= \int_{\partial M} fU \cdot \hat{n} \, dx && \text{(divergence theorem)} \\ &= 0 && \text{(since } U \cdot \hat{n} = 0\text{)}\end{aligned}$$

So the Weyl decomposition is orthogonal in L^2. In particular, any vector field orthogonal to every divergence-free field must be a gradient. Thus $\eta_{tt} = -\nabla p \circ \eta$ as desired.

We can also write

$$P_\eta(\eta_{tt}) = 0$$

where the projection operator is $P_\eta(V \circ \eta) = P(V) \circ \eta$.

Geometrically we can think of this as the following:

- $t \mapsto \eta(t)$ is a curve in the space of volumorphisms, so we can differentiate it in the standard way.
- But $\frac{d\eta}{dt}$ is a vector field along the curve. We can't differentiate this, because we can't slide vectors around a curved space.
- But we can define the covariant derivative of a vector field V along the curve η as $\dfrac{DV}{dt} = P_\eta\left(\dfrac{dV}{dt}\right)$
- (Take the derivative in the space of *all* diffeomorphisms, then project parallel to the space of volumorphisms.)

This works quite generally for curved surfaces in flat space; it's called the Levi-Civita derivative.

An alternate way to think of the Euler equation is to write

$$\frac{d^2\eta}{dt^2} = B\left(\frac{d\eta}{dt}, \frac{d\eta}{dt}\right)$$

where we define

$$B(X \circ \eta, Y \circ \eta) = \nabla \Delta^{-1} \operatorname{div}(X \cdot \nabla Y) \circ \eta$$

In geometry, B is called the *second fundamental form*: it takes two vectors parallel to the manifold and gives a vector perpendicular. It's symmetric...

$$\operatorname{div}(X \cdot \nabla Y) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x_j}\left(X_k \frac{\partial Y_j}{\partial x_k}\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial X_k}{\partial x_j} \frac{\partial Y_j}{\partial x_k}$$

since $\operatorname{div} Y = 0$.

The second fundamental form gives a measure of how a surface curves in a flat ambient space.

A remarkable property of this geodesic equation is that it makes sense as an ordinary differential equation in an infinite-dimensional space.

Usually, trying to write a PDE as an ODE in infinite-dimensional space leads to equations like

$$u_{tt} = F(u)$$

where F is not continuous in any reasonable topology.

In this case, the operator $\dot{\eta} \mapsto B(\dot{\eta}, \dot{\eta})$ is not only continuous, but smooth, in a strong Sobolev topology.

Loosely speaking, if η and $\dot{\eta} = X \circ \eta^{-1}$ are Sobolev H^s in the spatial coordinates (i.e., all derivatives up to order s are in L^2), then $\operatorname{div}(X \cdot \nabla X)$ is H^{s-1}.

Therefore

$$B(\dot{\eta}, \dot{\eta}) = \nabla \Delta^{-1} \operatorname{div}(X \cdot \nabla X) \circ \eta$$

is in H^s. It's much harder, but possible, to prove this operator is smooth in the H^s topology.

So the geodesic equation is a well-behaved ODE in the H^s topology, which gives local existence and uniqueness. This only works for s large, though. If s is too small, then we have to worry about the product of H^s functions still being in H^s. Thus we do *not* get a well-behaved ODE in the "natural" L^2 topology.

Now we want to linearize the geodesic equation.

Consider a variation $\eta(t,s)$ of a geodesic $t \mapsto \eta(t)$ in the space of volumorphisms. Set $J(t) = \partial_s\big|_{s=0}\eta(t,s)$. Then $Y(t) = J(t) \circ \eta(t)^{-1}$ is a vector field on M. Let us also set $Z(t) = \partial_s\big|_{s=0}X(t,s)$.

Recall the geodesic equation can be written as the pair of equations

$$\begin{aligned} \frac{\partial \eta}{\partial t} - X \circ \eta = 0 \qquad & \text{flow equation} \\ \frac{\partial X}{\partial t} + P(X \cdot \nabla X) = 0 \qquad & \text{Euler equation.} \end{aligned}$$

$$\text{Flow: } \frac{\partial \eta}{\partial t} - X \circ \eta = 0$$

If we differentiate the flow equation with respect to s, we obtain

$$\begin{aligned} \frac{\partial^2 \eta}{\partial t \partial s} - Z \circ \eta - (Y \cdot \nabla X) \circ \eta &= 0 \\ \frac{\partial}{\partial t}(Y \circ \eta) - Z \circ \eta - (Y \cdot \nabla X) \circ \eta &= 0 \\ \frac{\partial Y}{\partial t} \circ \eta + (X \cdot \nabla Y) \circ \eta - Z \circ \eta - (Y \cdot \nabla X) \circ \eta &= 0 \end{aligned}$$

Thus the linearized flow equation is

$$\frac{\partial Y}{\partial t} + [X, Y] = Z,$$

which can be written in terms of the covariant derivative as

$$\frac{DJ}{dt} - P(\nabla_J X) = Z \circ \eta.$$

$$\text{Euler: } \frac{\partial X}{\partial t} + P(X \cdot \nabla X) = 0$$

If we differentiate the Euler equation with respect to s, we obtain

$$\frac{\partial Z}{\partial t} + P(X \cdot \nabla Z) + P(Z \cdot \nabla X) = 0,$$

and in terms of the covariant derivative as

$$\frac{D}{dt}(Z \circ \eta) + P(Z \cdot \nabla X) \circ \eta = 0$$

Combining the linearized Euler equation and the linearized flow equation, we get a second-order equation for $J = Y \circ \eta$:

$$\begin{aligned}\frac{D^2}{dt^2}(Y \circ \eta) &= \frac{D}{dt} P(Y \cdot \nabla X) \circ \eta - P(Z \cdot \nabla X) \circ \eta \\ &= \frac{\partial}{\partial t} P(Y \cdot \nabla X) \circ \eta + P\big(\nabla_X P(Y \cdot \nabla X)\big) \circ \eta \\ &\qquad - P(\nabla_{\partial_t Y} X) \circ \eta - P([X, Y] \cdot \nabla X) \circ \eta \\ &= P\Big(- Y \cdot \nabla P(X \cdot \nabla X) + X \cdot \nabla P(Y \cdot \nabla X) \\ &\qquad - [X, Y] \cdot \nabla X\Big) \circ \eta\end{aligned}$$

The operator that appears on the right hand side is called the *Riemann curvature operator*:

$$R(J, \dot{\eta})\dot{\eta} = -P\Big(-Y \cdot \nabla P(X \cdot \nabla X) + X \cdot \nabla P(Y \cdot \nabla X) - [X, Y] \cdot \nabla X\Big) \circ \eta.$$

The linearized geodesic equation is called the *Jacobi equation*:

$$\frac{D^2 J}{dt^2} + R(J, \dot{\eta})\dot{\eta} = 0$$

If we are considering geodesics starting from the same point with very close initial directions, then $J(0) = 0$.

We should think of the Jacobi equation intuitively as

$$\frac{d^2 j}{dt^2} + cj(t) = 0$$

The solutions of this with $j(0) = 0$ are $j(t) = j'(0)S_c(t)$ where

$$S_c(t) = \begin{cases} \frac{1}{\sqrt{c}} \sin\left(\sqrt{c}t\right) & c > 0 \\ t & c = 0 \\ \frac{1}{\sqrt{|c|}} \sinh\left(\sqrt{|c|}t\right) & c < 0 \end{cases}$$

With this analogy in mind, we define the *sectional curvature* by

$$K(J,\dot\eta) = \frac{\langle R(J,\dot\eta)\dot\eta, J\rangle_{L^2}}{\langle J, J\rangle_{L^2}}$$

if J and $\dot\eta$ are L^2-orthogonal and $\|\dot\eta\|_{L^2} = 1$.

Of course the Jacobi equation is much more complicated than the constant-coefficient model. However the *Rauch comparison theorem* (perhaps the most useful result in Riemannian geometry) allows us to compare solutions of the Jacobi equation to the constant-coefficient model if we have an inequality on the sectional curvature.

Theorem (Rauch)**.** *If $K(J,\dot\eta) \le k$ for every J orthogonal to $\dot\eta$ and for every t, then the solution of the Jacobi equation satisfies $|J(t)|_{L^2} \ge |J'(0)|_{L^2} S_k(t)$ up to the first vanishing point of $S_k(t)$.*

Some geodesic deviations for simple surfaces.

The original idea of Vladimir Arnold (who pioneered this geometric approach to ideal fluid mechanics) was:

- Show that the sectional curvature is always negative for certain flows.
- Use the Rauch comparison theorem to show that Jacobi fields grow exponentially in time.
- Identify Jacobi fields as linearized Lagrangian perturbations (perturbations of the particle paths, not of the velocity field).
- Conclude that such flows are unstable in the Lagrangian sense (i.e., particle paths diverge exponentially).

The problem is that there are *no* known flows for which the sectional curvature is always less than a negative constant. It seems likely there are none.

Negative curvature is very common.

Theorem (Rouchon, Misiolek). *The only way a geodesic of volumorphisms can have* $K(J, \dot{\eta}) \geq 0$ *for every* J *is if* η *is a family of rotations.*

However *uniformly* negative curvature is rare.

Theorem (P). *The only ways a two-dimensional steady solution of the Euler equation* $P(\nabla_X X) = 0$ *can have* $K(J, \dot{\eta}) \leq 0$ *for every* J *are if* $X = u(r)\mathbf{e}_\theta$ *(rotationally symmetric) or* $X = u(ax + by)(-b\mathbf{e}_x + a\mathbf{e}_y)$ *(unidirectional).*

Zero curvature is unavoidable, and often prevents exponential growth.

A formula for the curvature:

$$\begin{aligned}
K(X,Y) &= -\int_M Y \cdot P\Big(-Y \cdot \nabla P(X \cdot \nabla X) + X \cdot \nabla P(Y \cdot \nabla X) \\
&\qquad\qquad - [X,Y] \cdot \nabla X \Big)\, dx \\
&= -\int_M Y \cdot \Big(-Y \cdot \nabla P(X \cdot \nabla X) + X \cdot \nabla P(Y \cdot \nabla X) \\
&\qquad\qquad - [X,Y] \cdot \nabla X \Big)\, dx \\
&= -\int_M Y \cdot \Big(-Y \cdot \nabla\nabla p_{XX} + X \cdot \nabla\nabla p_{XY} \Big)\, dx \\
&= -\int_M (Y \cdot \nabla Y) \cdot (\nabla p_{XX}\, dx + \int_M (X \cdot \nabla Y) \cdot \nabla p_{XY}\, dx \\
&= \int_M \nabla p_{YY} \cdot \nabla p_{XX}\, dx - \int_M \nabla p_{XY} \cdot \nabla p_{XY}\, dx
\end{aligned}$$

Notice that $\nabla p_{XY} = B(X, Y)$, the orthogonal component of $X \cdot \nabla Y$. This is a general formula:

$$K(X, Y) = \langle B(X, X), B(Y, Y)\rangle - \langle B(X, Y), B(X, Y)\rangle.$$

What's interesting about this is that $B(X, Y)$ depends on how a surface sits in a flat ambient space.

For example, a cone has nonvanishing second fundamental form, while a plane has vanishing second fundamental form. Intrinsically these surfaces are the same (if you were living on a portion of a cone, you'd think it was a plane).

The sectional curvature *does* distinguish between geometrically different surfaces. (We can tell that earth is curved even in a small portion of it, just by drawing geodesics in it.)

Gauss called this fact his "Remarkable Theorem." Since Riemann, much of geometry has been focused on intrinsic properties.

So we see that the geometric approach leads to:

- Nice local existence and uniqueness results for the PDE.
- Criteria for stability (in the Lagrangian sense).

We can also look at other properties. For example, blowup of the Euler equation solutions (in three dimensions) may be related to geodesics meeting very often. Thus we have been researching how geodesics can separate and come together in the space of volumorphisms.

The same approach can be taken with many other equations of continuum mechanics. For example:

- the motion of an inextensible string
- the KdV equation
- the equations of plasma physics

There is thus a lot to do...