Math 4001 Analysis 2
Homework Set 7 (Take Home Final)

Spring 2009

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Problem 1: Prove that the function

$$F : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}, \quad (x,t) \mapsto t^{-n/2} \exp\big(-\|x\|^2/4t\big)$$

satisfies the differential equation

$$\Delta F - \frac{\partial F}{\partial t} = 0,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is the so-called Laplacian on $\mathbb{R}^n$. (8P)

Problem 2: Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) := \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{for } (x,y) \neq (0,0), \\ 0, & \text{for } (x,y) = (0,0). \end{cases}$$

Prove that g is twice partial differentiable, but that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} g(0,0) \neq \frac{\partial}{\partial y}\frac{\partial}{\partial x} g(0,0).$$

Is g continuous at the origin? (8P)

Problem 3: Compute the Jacobian of the function

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (r,\theta,\varphi) \mapsto (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).$$

(6P)

Problem 4: Compute the divergence of the vector field

$$F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n, \quad x \mapsto \frac{x}{\sqrt{x_1^2 + \ldots + x_n^2}}.$$

(6P)

Problem 5: Determine the Taylor expansion of the function

$$h : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}, \quad h(x,y) = \frac{x-y}{x+y}$$

at the point $(1,1)$ up to order 2. (6P)