Math 6220 Introduction to Topology 2
Homework Set 2
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Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 204, Telephone: 2-7717, e-mail: markus.pflaum@colorado.edu.

By a semi-cosimplicial module (over a given ring $R$) one understands a sequence of $R$-modules $M^k$, $k \in \mathbb{N}$ together with $R$-module maps $d^i : M^k \to M^{k+1}$, $(0 \leq i \leq k+1)$ such that

$$d^j d^i = d^i d^{j-1} \text{ for } i < j.$$ 

Show that then $(M^*, d)$ with $d = \sum_{i=0}^{k+1} (-1)^i d^i$ is a cochain complex. Its cohomology groups will be denoted by $H^k(M)$.

**Singular cohomology.** Let $X$ be a topological space, and $k$ a unital commutative ring. Put $S^k(X) := \text{Hom}_\mathbb{Z}(S_k(X), \mathbb{Z})$, and let $d^*\ k$ be dual to the face map $\delta_i : S_k(X) \to S_{k-1}(X)$. Prove that one thus obtains a semi-cosimplicial module. The corresponding cohomology is called singular cohomology of $X$ with coefficients on $\mathbb{Z}$.

**Čech cohomology.** Let $X$ be a topological space, and $\mathcal{U}$ an open covering of $X$. Denote for each $k$ by $U(k)$ the set of all $(k+1)$-tupels $(U_0, \ldots, U_k)$ of elements of $\mathcal{U}$ such that $U_0 \cap \ldots \cap U_k \neq \emptyset$. For each $k \in \mathbb{N}$ let

$$\check{C}^k(X) := \prod_{(U_0, \ldots, U_k) \in \mathcal{U}(k)} \mathbb{R}(U_0 \cap \ldots \cap U_k),$$

where for an open set $U \subset X$ one denotes by $\mathbb{R}(U)$ the space of locally constant functions from $U$ to $\mathbb{R}$. Define now $\check{\delta}_i : \check{C}^k(X) \to \check{C}^{k+1}(X)$ by

$$\left((f(U_0, \ldots, U_k))_{(U_0, \ldots, U_k) \in \mathcal{U}(k)} \mapsto (f(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{k+1}))_{(U_0, \ldots, U_{i+1}) \in \mathcal{U}(k+1)} \right).$$

Check that one thus obtains a semi-cosimplicial space; its cohomology is called the Čech cohomology of $X$ with respect to the covering $\mathcal{U}$ and is denoted by $\check{H}^*(X)$.

**Alexander–Spanier cohomology.** Let $X$ be a topological space. For $k \in \mathbb{N}$ denote by $C^k(X)$ the space of all continuous functions $f$ from $X^{k+1}$ to $\mathbb{R}$. Define face maps $\delta^i : C^k(X) \to C^{k+1}(X)$ by the formula

$$\delta^i f(x_0, \ldots, x_{k+1}) = f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}), \quad f \in C^k(X).$$

Show that one thus obtains a semi-cosimplicial space. The resulting cochain complex $(C^*(X), \delta)$ is acyclic (10 extra points for a proof of this!) By passing to an appropriate quotient complex one obtains a complex which has nontrivial cohomology and contains the information one wants.

To define that quotient complex consider the subcomplex $C_0^*(X)$ of locally vanishing cochains. In degree $k$, this complex consists of all $f \in C^k(X)$ for which there exists an open covering $\mathcal{U}$ of $X$ such that $f$ vanishes on the neighborhood $U^{(k+1)} = \bigcup_{U \in \mathcal{U}} U^{k+1}$. Show that $\delta f$ is locally vanishing, if $f$ is. The quotient complex $(C^k_{\text{AS}}(X), \delta) := (C^*(X)/C_0^*(X), \delta)$ is called the Alexander–Spanier complex of $X$ (with coefficients in $\mathbb{R}$), and its cohomology the Alexander–Spanier cohomology of $X$. One denotes the Alexander–Spanier cohomology of $X$ by $H^*_{\text{AS}}(X)$. 