AN ALGEBRAIC INDEX THEOREM FOR ORBIFOLDS

M.J. PFLAUM, H.B. POSTHUMA AND X. TANG

Abstract. Using the concept of a twisted trace density on a cyclic groupoid, a trace is constructed on a formal deformation quantization of a symplectic orbifold. An algebraic index theorem for orbifolds follows as a consequence of a local Riemann–Roch theorem for such densities. In the case of a reduced orbifold, this proves a conjecture by Fedosov, Schulze, and Tarkhanov. Finally, it is shown how the Kawasaki index theorem for elliptic operators on orbifolds follows from this algebraic index theorem.

Contents

Date: May 5, 2006.

Introduction

Index theory originated with the seminal paper [AtSi] of Atiyah and Singer almost 40 year ago. They proved that the index of an elliptic operator on a closed Riemannian manifold M depends only on the class of the principal symbol in the K-theory of T^*M. Ever since, many new proofs and generalizations of this theorem have appeared. To mention, the index theorem has been extended to the equivariant case, to families of operators, and to foliations.

The generalization this paper is concerned with is the index theorem for formal deformation quantizations originally proved by Fedosov [Fe96] and (independantly) by Nest–Tsygan [NeTs95]. This theorem, which in the literature is known as the algebraic index theorem, is in principle an abstract result computing the pairing of K-theory classes with the cyclic cocycle given by the unique trace on a formal deformation quantization of a symplectic manifold. However, as shown in [NeTs96], the nomenclature "index theorem" may be justified by the fact that in the case of the cotangent bundle with its canonical symplectic form and the deformation quantization given by the asymptotic pseudo-differential calculus, one recovers the original Atiyah–Singer index theorem.

Here, we prove an algebraic index theorem for formal deformation quantizations of symplectic orbifolds and derive from it its analytic version, the well-known Kawasaki index formula for orbifolds. Let us explain and state the theorem in some more detail.

In our setup, see Section 1 for more details, a symplectic orbifold X is modeled by a proper étale groupoid $G : G_1 \rightrightarrows G_0$ with an invariant, nondegenerate two-form on G_0. Consequently, we consider formal deformation quantizations of the convolution algebra of G. As in [Ta], these are constructed by a crossed product construction of a G-invariant formal deformation $\mathcal{A}^\hbar$ of G_0 by G, denoted $\mathcal{A}^\hbar \rtimes G$. This is the starting point for the algebraic index theorem.

By computing the cyclic cohomology of the algebra $\mathcal{A}^\hbar \rtimes G$ we have given a complete classification of all traces in our previous paper [NePfPoTa]: the dimension of the space of traces equals the number of connected components of the so-called inertia orbifold $\tilde{X}$ associated to X. Therefore, in contrast to the case of symplectic manifolds, there is no unique (normalized) trace on the deformation of the convolution algebra of G. In this paper, we construct a particular trace Tr using the notion of a twisted trace density, the appropriate generalization of a trace density to orbifolds. Standard constructions in K-theory, see Section 1, yield a map

$$\mathrm{Tr}_* : K^0_{\mathrm{orb}}(X) \to \mathbb{C}((\hbar)),$$

associated to the trace Tr, called the index map. Here, $K^0_{\mathrm{orb}}(X)$ is the Grothendieck group generated by isomorphism classes of so called orbifold vector bundles. The index theorem proved in this paper expresses the value of this map on a virtual orbifold vector bundle $[E] - [F]$ in terms of the characteristic classes of E, F, the orbifold X and the chosen deformation.

Theorem *Let G be a proper étale Lie groupoid representing a symplectic orbifold X. Let E and F be G-vector bundles which are isomorphic outside a compact subset of X. Then the following formula holds for the index of $[E] - [F]$:*

$$\mathrm{Tr}_*([E] - [F]) = \int_{\tilde{X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta\left(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i}\right)}{\det\left(1 - \theta^{-1}\exp\left(-\frac{R^\perp}{2\pi i}\right)\right)} \hat{A}\Big(\frac{R^T}{2\pi i}\Big) \exp\Big(-\frac{\iota^*\Omega}{2\pi i\hbar}\Big).$$

In this formula, the right hand side is a purely topological expression that we now briefly explain: Let B_0 be the space of objects of the groupoid modeling the inertia orbifold $\tilde{X}$, and $\iota : B_0 \to G_0$ the canonical map explained in Section 1. This groupoid has a canonical cyclic structure θ acting on the fibers the vector bundles E and F as well as the normal bundle to the map ι. This enables one to define the twisted Chern character Ch_θ, see Section 5. The coefficient function $\frac{1}{m}$ is defined in terms of the order of the local isotropy groups (see Sections 1.3 and 4.3). The factor $\hat{A}(\ldots)$ is the standard characteristic class of the symplectic manifold B_0 associated to the bundle of symplectic frames, and the factor $\det(\ldots)$ is the inverse of the twisted Chern character associated to the symplectic frame bundle on the normal bundle to $\iota : B_0 \to G_0$. Finally, Ω is the characteristic class of the deformation quantization of G_0.

Let us make several remarks about this Theorem. First, indeed notice that the right hand side is a formal Laurent series in $\mathbb{C}((\hbar))$. Only when the characteristic class of the deformation quantization is trivial, it is independent of $\hbar$. This happens for example in the case of a cotangent bundle T^*X to a reduced orbifold X, with the canonical deformation quantizatioin constructed from asymptotic pseudodifferential calculus, in which case we show in Section 6 that the left hand side equals the index of an elliptic operator on X. This is exactly the Kawasaki index theorem [Ka], originally derived from the Atiyah–Segal–Singer G-index theorem. An alternative proof using operator algebraic methods has been given by Farsi [Fa].

Notice that, since we work exclusively with the convolution algebra of the groupoid and its deformation, the abstract theorem above also holds for nonreduced orbifolds. In the reduced case it proves a conjecture of Fedosov–Schulze–Tarkhanov [FeSchTa]. Although they work with the different algebra of invariants of $\mathcal{A}^\hbar(G_0)$ instead of the crossed product, one can show that in the reduced case the two are Morita equivalent, allowing for a precise translation.

The methods used to prove our main result are related to the proof of the algebraic index theorem on a symplectic manifold in [FeFeSh]. One of the main tools in that paper is the construction of a trace density for a deformation quantization on the underlying symplectic manifold. We generalize the construction of such a trace density to symplectic orbifolds. Hereby, our approach is inspired by the localization behavior of cyclic cocycles on the deformed algebra, which has been discovered in our previous paper [NePfPoTa], and includes an essential new feature, a local "twisting" on the inertia groupoid.

Let us finally mention that by the local nature of the proof of our index theorem an even stronger results holds true, namely a local algebraic index formula for orbifolds. The precise statement of this is given in Theorem 6.2.

Our paper is set up as follows. After introducing some preliminary material in Section 1, we introduce in Setion 2 the concept of a twisted trace density. In Section 3, we construct certain local twisted Hochschild cocyles, which in Section 4 will be "glued" to a twisted trace density. Thus, we obtain a trace on the deformation quantization of the groupoid algebra, and consequently the index map on $K^0_{\mathrm{orb}}(X)$. In Section 5, we use Chern-Weil theory on Lie algebras to determine the cohomology class of the trace density defined in Section 4, and obtain a local algebraic Riemann-Roch formula for symplectic orbifolds. Finally, in Section 6 we prove the above theorem and use it to give an algebraic proof of the classical

Kawasaki index theorem for orbifolds. In the Appendix, we provide some material needed for the proof of our index theorem. More precisely, in Appendix A we determine several Lie algebra cohomologies and in Appendix B we explain the asymptotic pseudodifferential calculus and its relation to deformation quantization.

Acknowledgement: M.P. and H.P. acknowledge financial support by the Deutsche Forschungsgemeinschaft. H.P. and X.T. would like to thank the Fachbereich Mathematik of Goethe-Universität at Frankfurt/Main for the hospitality during their visits. X.T. would like to thank Dmitry Fuchs, Ilya Shapiro, and Alan Weinstein for helpful suggestions. H.P. is supported by EC contract MRTN-CT-2003-505078 (LieGrits).

1. Preliminaries

In this section we briefly recall some of the essential points of [NePfPoTa]. For any orbifold X, we choose a proper étale Lie groupoid $G_1 \rightrightarrows G_0$ for which $G_0/G_1 \cong X$. Such a groupoid always exists and is unique up to Morita equivalence, although being étale is not Morita invariant. The groupoid is used to describe the differential geometry of the orbifold. For example, an orbifold vector bundle is equivalent to a G-vector bundle, that is, a vector bundle $E \to G_0$ with an isomorphism $s^*E \cong t^*E$. The notion of a G-sheaf is similarly defined.

The Burghelea space of G, also called "space of loops", is defined by

$$B_0 = \{g \in G_1 \mid s(g) = t(g)\},$$

where $s, t : G_1 \to G_0$ are the source and target map of the groupoid G. Denote by ι the canonical embedding $B_0 \hookrightarrow G_1$. The groupoid G acts on B_0 by conjugating the loops and this defines the associated inertia groupoid $\Lambda G := B_0 \rtimes G$. This groupoid turns out also to be proper and étale, and therefore models an orbifold, $\widetilde{X}$, called the inertia orbifold.

An important property of the inertia groupoid which will be essential in this paper is the existence of a *cyclic structure* [Cr]: there is a canonical section $\theta : \Lambda G_0 \to \Lambda G_1$ of both the source and target maps of ΛG, given by $g \mapsto \theta_g$. Here $\theta_g = g$, viewed as a morphism from g to g.

The convolution algebra of G is defined as the vector space $\mathcal{C}_{\mathrm{c}}^\infty(G_1)$ with the following product:

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2), \quad f_1, f_2 \in \mathcal{C}_{\mathrm{c}}^\infty(G_1),\ g \in G_1. \tag{1.1}$$

The convolution algebra is denoted by $\mathcal{A} \rtimes G$, where $\mathcal{A}$ here and everywhere in this article means the G-sheaf of smooth functions on G_0.

When the orbifold X is symplectic, one can choose G in such a way that G_0 carries a symplectic form ω which is invariant, i.e., $s^*\omega = t^*\omega$. For such a groupoid, we choose a G-invariant deformation quantization of G_0, giving rise to a G-sheaf of algebras $\mathcal{A}^\hbar$ on G. The crossed product algebra $\mathcal{A}^\hbar \rtimes G$ defines a formal deformation quantization of the convolution algebra $\mathcal{A} \rtimes G$ with its canonical noncommutative Poisson structure induced by ω. Recall that $\mathcal{A}^\hbar \rtimes G$ is defined as the vector space $\Gamma_{\mathrm{c}}(G_1, s^*\mathcal{A}^\hbar)$ with product

$$[a_1 \star_c a_2]_g = \sum_{g_1\, g_2 = g} \big([a_1]_{g_1} g_2\big)[a_2]_{g_2}, \quad a_1, a_2 \in \Gamma_{\mathrm{c}}(G_1, s^*\mathcal{A}^\hbar),\ g \in G, \tag{1.2}$$

where $[a]_g$ denotes the germ of a at g. Closely related is the algebra $\Gamma_{\mathrm{inv,c}}(\mathcal{A}^\hbar)$ of G-invariant sections, first considered in [Pf98]. In fact, when X is reduced, the two are Morita invariant, cf [NePfPoTa, Prop. 6.5.], the equivalence bimodule being given by $\mathcal{A}^\hbar_c(G_0)$.

Let us finally mention that for every G-sheaf $\mathcal{S}$ on G_0, we will denote the sheaf $s^{-1}\mathcal{S} = t^{-1}\mathcal{S}$ also by $\mathcal{S}$. Since G is assumed to be proper étale, this will be convenient and not lead to any misunderstandings.

1.1. **Traces on $\mathcal{A}^\hbar \rtimes G$.** The starting point of the present article is the classification of traces on the deformed convolution algebra $\mathcal{A}^\hbar \rtimes G$ in [NePfPoTa], which we now briefly recall. A trace on the algebra $\mathcal{A}^\hbar \rtimes G$ is an $\hbar$-adically continuous linear functional $\mathrm{Tr} : \mathcal{A}^\hbar \rtimes G \to \mathbb{K}$, where $\mathbb{K}$ denotes the field of Laurent series $\mathbb{C}((\hbar))$, such that

$$\mathrm{Tr}(a \star_c b) = \mathrm{Tr}(b \star_c a).$$

Of course, a trace on an algebra is nothing but a cyclic cocycle of degree 0. Moreover, the space of traces on $\mathcal{A}^\hbar \rtimes G$ is in bijective correspondence with the space of traces on the extended deformed convolution algebra $\mathcal{A}^{((\hbar))} \rtimes G$, where $\mathcal{A}^{((\hbar))} = \mathcal{A}^\hbar \otimes_{\mathbb{C}[[\hbar]]} \mathbb{K}$. One of the main results of [NePfPoTa] now asserts that

$$HC^p(\mathcal{A}^{((\hbar))} \rtimes G) \cong \bigoplus_{l \geq 0} H^{p-2l}\left(\tilde{X}, \mathbb{K}\right), \tag{1.3}$$

and therefore HC^0 equals $H^0(\tilde{X}, \mathbb{C}) \otimes \mathbb{K}$. From this it follows that the number of linear independent traces on $\mathcal{A}^\hbar \rtimes G$ equals the number of connected components of the inertia orbifold $\tilde{X}$. In [NePfPoTa], a construction of all these traces was given using a Čech-like description of cyclic cohomology, however the resulting formulas are not easily applicable to index theory. Therefore, we start in Section 2 by giving an alternative, more local, construction of traces on deformed groupoid algebras using the notion of a twisted trace density.

1.2. **The index map.** Here we briefly explain the construction of the index map, given a trace $\mathrm{Tr} : \mathcal{A}^\hbar \rtimes G \to \mathbb{K}$ on the deformed convolution algebra. As is well known, a trace $\tau : A \to \Bbbk$ on an algebra A over a field $\Bbbk$ induces a map in K-theory $\tau_* : K_0(A) \to \Bbbk$ by taking the trace of idempotents in $M_n(A)$. In our case, we obtain a map

$$\mathrm{Tr}_* : K_0(\mathcal{A}^{((\hbar))} \rtimes G) \to \mathbb{C}((\hbar)).$$

The inclusion $\mathcal{A}^\hbar \rtimes G \hookrightarrow \mathcal{A}^{((\hbar))} \rtimes G$ induces a map $K_0(\mathcal{A}^\hbar \rtimes G) \to K_0(\mathcal{A}^{((\hbar))} \rtimes G)$. Since $\mathcal{A}^\hbar \rtimes G$ is a deformation quantization of the convolution algebra of G, one has, by rigidity of K-theory

$$K_0(\mathcal{A}^\hbar \rtimes G) \cong K_0(\mathcal{A} \rtimes G) \cong K^0(G).$$

Here, $K^0(G)$ is the Grothendieck group of isomorphism classes of G-vector bundles on G_0, also called the orbifold K-theory $K^0_{\mathrm{orb}}(X)$. To be precise, the isomorphism $K_0(\mathcal{A} \rtimes G) \cong K^0(G)$, supposing that X is compact, associates to any G-vector bundle $E \to G_0$ the projective $\mathcal{C}^\infty_{\mathrm{c}}(G)$-module $\Gamma_{\mathrm{c}}(E)$, where $f \in \mathcal{C}^\infty_{\mathrm{c}}(G)$ acts on $s \in \Gamma_{\mathrm{c}}(E)$ by

$$(f \cdot s)(x) = \sum_{t(g)=x} f(g)\, s(x), \text{ for } x \in G_0.$$

Putting all these maps together, the trace Tr defines a map

$$\mathrm{Tr}_* : K^0_{\mathrm{orb}}(X) \to \mathbb{C}((\hbar)). \tag{1.4}$$

As we have seen in the previous paragraph, traces on $\mathcal{A}^{((\hbar))} \rtimes G$ are highly non-unique on general orbifolds, and all induce maps as in (1.4). The trace we will construct in this paper using the index density has the desirable property that it has support on each of the components of $\tilde{X}$. Consequently, we will refer to the map (1.4) induced by this trace as the index map and the main theorem proved in this paper gives a cohomological formula for the value on a given orbifold vector bundle E. Notice that because of the support properties of this "canonical trace", this theorem in principle solves the index problem for any other element in $HC^0(\mathcal{A}^{((\hbar))} \rtimes G)$.

1.3. **Integration on orbifolds.** Let Γ be a finite group acting by diffeomorphims on a smooth manifold M of dimension n. Consider the orbifold $X = M/\Gamma$, and assume X to be connected. Recall that X carries a natural stratification by orbit types (see [Pf01, Sec. 4.3]). Denote by X° the principal stratum of X and by $M^\circ \subset M$ its preimage under the canonical projection $M \to X$. Now let μ be an n-form on X or in other words a Γ-invariant n-form on M. The integral $\int_X \mu$ then is defined by

$$\int_X \mu := \frac{m}{|\Gamma|} \int_M \mu, \tag{1.5}$$

where $m \in \mathbb{N}$ is the order of the isotropy group Γ_x of some point $x \in M^\circ$. Observe that by the slice theorem m does not depend on the particular choice of x, since X is connected. Let us mention that formula (1.5) is motivated by the fact that on the one hand the covering $M^\circ \to X^\circ$ has exactly $\frac{|\Gamma|}{m}$ sheets and on the other hand the singular set $X \setminus X^\circ$ has measure 0.

Assume now to be given an arbitrary orbifold X. To define an integral over X choose a covering of X by orbifold charts and a subordinate smooth partition of unity. Locally, the integral is defined by the above formula for orbit spaces. These local integrals are glued together globally by the chosen partition of unity. The details of this construction are straightforward.

2. Twisted trace densities

Since $\mathcal{A}^\hbar$ is a G-sheaf of algebras, the pull-back sheaf $\iota^{-1}\mathcal{A}^\hbar$ is canonically a ΛG-sheaf. This implies that every stalk $\iota^{-1}\mathcal{A}^\hbar_g$, $g \in B_0$ has a canonical automorphism given by the action of the cyclic structure $\theta_g \in \mathrm{Aut}(\iota^{-1}\mathcal{A}^\hbar_g)$. Alternatively, θ defines a canonical section of the sheaf $\underline{\mathrm{Aut}}(\iota^{-1}\mathcal{A}^\hbar)$ of automorphisms of $\iota^{-1}\mathcal{A}^\hbar$. This facilitates the following definition.

Definition 2.1. A θ-twisted trace density on a ΛG-sheaf of algebras $\iota^{-1}\mathcal{A}^\hbar$ is a sheaf morphism $\psi : \iota^{-1}\mathcal{A}^\hbar \to \Omega^{\mathrm{top}}_{\Lambda G}$ which satisfies

$$\psi(ab) - \psi(\theta(b)a) \in d\Omega^{\mathrm{top}-1}_{\Lambda G}. \tag{2.1}$$

In this definition $\Omega^{\mathrm{top}}_{\Lambda G}$ means the sheaf of top degree de Rham forms on every connected component of B_0. Therefore, the integral of a twisted trace density over B_0 is well defined. Notice that although $\Omega^\bullet_{\Lambda G}$ is a ΛG-sheaf, the action of θ is trivial, and therefore the twisting in the definition only involves $\iota^{-1}\mathcal{A}^\hbar$. As we will see later this involves the normal bundle to the embedding $\iota : B_0 \hookrightarrow G_1$.

Proposition 2.2. *When ψ is a θ-twisted trace density, the formula*

$$\mathrm{Tr}(a) = \int_{B_0} \psi(a_{|B_0}), \quad a \in \mathcal{A}^{((\hbar))} \rtimes G,$$

defines a trace on the deformed convolution algebra $\mathcal{A}^{((\hbar))} \rtimes G$.

Proof. The map $a \mapsto a_{|B_0}$ is the degree 0 part of a natural morphism

$$C_\bullet(\mathcal{A}^{((\hbar))} \rtimes G) \to \Gamma_{\mathrm{c}}\left(B_\bullet, \sigma^{-1}\mathcal{A}^{((\hbar))}_{G_0^{\bullet+1}}\right),$$

of complexes called "reduction to loops". Using the notation from [NePfPoTa, Sec. 2.6]), $\mathcal{A}^{((\hbar))}_{G_0^p}$ denotes here the sheaf $(\mathcal{A}^{((\hbar))})^{\boxtimes p}$ on the cartesian product G_0^p, B_p the Burghelea space given by

$$B_p = \{(g_0, \cdots, g_p) \in G^{p+1} \mid t(g_0) = s(g_p), \cdots, t(g_p) = s(g_{p-1})\},$$

and $\sigma : B_p \to G_0^{p+1}$ the map $(g_0, \cdots, g_p) \mapsto (s(g_0), \cdots, s(g_p))$. Moreover, the differential on $C_\bullet$ is the standard Hochschild differential on $\mathcal{A}^{((\hbar))} \rtimes G$, and the differential on the right hand side is given in [Cr, NePfPoTa]. In the following, we will only need the first one

$$d_0 - d_1 : \Gamma_{\mathrm{c}}\left(B_1, \sigma^{-1}\mathcal{A}^{((\hbar))}_{G_0^2}\right) \to \Gamma_{\mathrm{c}}\left(B_0, \iota^{-1}\mathcal{A}^{((\hbar))}_{G_0}\right). \tag{2.2}$$

At the level of germs it is given by

$$\begin{aligned} d_0[a_0, a_1]_{(g_0,g_1)} &= [a_0 g_1 a_1]_{g_0 g_1}, \\ d_1[a_0, a_1]_{(g_0,g_1)} &= [a_1 g_0 a_0]_{g_1 g_0}. \end{aligned}$$

The complex $(C_\bullet(A), b)$ calculates the Hochschild homology $HH_\bullet(A)$ for any H-unital algebra A. Since $HH_0(A) = A/[A, A]$, a trace on A is nothing but a linear functional on $HH_0(A)$.

It was proved in [NePfPoTa, Prop. 5.7] that reduction to loops induces a quasi-isomorphism of complexes, in particular commutes with the differentials. Therefore, to construct a trace on $\mathcal{A}^\hbar \rtimes G$, it suffices to construct a linear functional on the vector space $\Gamma_{\mathrm{c}}(B_0, \iota^{-1}\mathcal{A}^{((\hbar))}_{G_0})$ which vanishes on the image of the map (2.2). ¿From the definition of the simplicial operators d_0 and d_1 above, we see that the germ at $g \in B_0$ of such a section is given by

$$\sum_{g_0 g_1 = g} \left([a_0] g_1 [a_1] - [a_1] g_1 g^{-1} [a_0]\right) = \sum_{g_0 g_1 = g} \left([a_0] g_1 [a_1]_{g_1} - \theta([a_1])_{g_1} [a_0])\right),$$

where, to pass over to the right hand side, we have used that $\mathcal{A}^{((\hbar))}$ is a G-sheaf of algebras. By the defining property of the twisted trace density such elements will be mapped into $d\Omega^{\mathrm{top}-1}_{B_0}$. Integrating, it follows from Stokes' theorem that the functional given by integrating the trace density vanishes on the image of $d_0 - d_1$. Combined with the restriction to B_0, it follows that the formula in the proposition defines a trace. □

Remark 2.3. A warning is in order here, as the formula in Proposition 2.2 for the trace appears to suggest that the trace only depends on the restriction of a formal power series $a \in \mathcal{C}^\infty_{\mathrm{c}}(G)[[\hbar]]$ to B_0. This is not true, since the reduction to loops maps an element a to its *germ* at B_0, viewed as an element of $\Gamma_{\mathrm{c}}(B_0, \iota^{-1}\mathcal{A}^\hbar)$, and otherwise the twisting condition would be trivial. In fact, we will see that the normal bundle to $B_0 \subset G_1$ plays an essential role in the computations below.

Concluding, to construct a trace, it suffices to construct a twisted trace density. Inspired by the recent construction in [FeFeSh] of a trace density from a certain Hochschild cocycle, we aim for a similar construction in a twisted Hochschild complex.

3. A twisted Hochschild cocycle

3.1. **Twisted Hochschild cohomology.** Let A be an algebra equipped with an automorphism $\gamma \in \mathrm{Aut}(A)$. There is a standard way to twist the Hochschild homology and cohomology of A by γ: recall that the Hochschild homology and cohomology $H_\bullet(A, M)$ and $H^\bullet(A, M)$ are defined with values in any bimodule M. Then one defines

$$HH^\gamma_\bullet(A) := H_\bullet(A, A_\gamma), \quad HH^\bullet_\gamma(A) := H^\bullet(A, A^*_\gamma), \tag{3.1}$$

where A_γ is the A-bimodule given by A with bimodule structure

$$a_1 \cdot a \cdot a_2 = a_1 a \gamma(a_2),\ a_1, a_2 \in A,\ a \in A_\gamma,$$

and A^*_γ denotes its dual. Clearly this definition is functorial with respect to automorphism preserving algebra homomorphisms. In the case of cohomology, let us write out the standard complex computing this twisted cohomology.

On the space of cochains $C^p(A) = \mathrm{Hom}(A^{\otimes p}, A^*) \cong \mathrm{Hom}(A^{\otimes(p+1)}, \mathbb{K})$, introduce the differential $b_\gamma : C^p \to C^{p+1}$ by

$$\begin{aligned}(b_\gamma f)(a_0 \otimes \cdots \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i f(a_0, \cdots, a_i a_{i+1}, \cdots, a_{p+1}) \\ + (-1)^{p+1} f(\gamma(a_{p+1}) a_0, a_1, \cdots, a_p).\end{aligned} \tag{3.2}$$

One checks that $b_\gamma^2 = 0$, and the cohomology of the resulting cochain complex is called the twisted Hochschild cohomology of A. When $\gamma = 1$, this definition reduces to the ordinary Hochschild cohomology $HH^\bullet(A)$.

3.2. **The local model.** Let $V = \mathbb{R}^{2n}$ equipped with the standard symplectic form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

in coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n}$. The Weyl algebra $\mathbb{W}_{2n}$ of $(\mathbb{R}^{2n}, \omega)$ is given over the field $\mathbb{K} = \mathbb{C}((\hbar))$ by the vector space $\mathbb{K}[p_1, \cdots p_n, q_1, \cdots, q_n]$ of polynomials in $(p_1, \cdots, p_n, q_1, \cdots, q_n)$ with the Moyal product defined by

$$a \star b = m\big(\exp \hbar\alpha(a \otimes b)\big).$$

Here, $m : \mathbb{W}_{2n} \otimes \mathbb{W}_{2n} \to \mathbb{W}_{2n}$ is the commutative product on polynomials and

$$\alpha(a \otimes b) = \sum_{i=1}^{n} \frac{\partial a}{\partial p_i} \otimes \frac{\partial b}{\partial q_i} - \frac{\partial b}{\partial p_i} \otimes \frac{\partial a}{\partial q_i}$$

denotes the action of the Poisson tensor on $\mathbb{W}_{2n} \otimes \mathbb{W}_{2n}$. The Weyl algebra is in fact a functor from symplectic vector spaces to unital algebras over $\mathbb{K}$, which implies that there is a canonical action of Sp_{2n}, the group of real linear symplectic transformations, on $\mathbb{W}_{2n}$ by automorphisms. This induces an action of $\mathfrak{sp}_{2n}(\mathbb{K}) := \mathfrak{sp}_{2n} \otimes_{\mathbb{R}} \mathbb{K}$, where $\mathfrak{sp}_{2n}$ is the Lie algebra of Sp_{2n}, by derivations. In fact, this action is inner, $\mathfrak{sp}_{2n}(\mathbb{K})$ being identified with the degree two homogeneous polynomials in $\mathbb{W}_{2n}$ acting by the commutator. In particular, a linear symplectic transformation

$\gamma \in \mathrm{Sp}_{2n}$ acts on the Weyl algebra by automorphisms and we can consider the twisted Hochschild homology of $\mathbb{W}_{2n}$.

Proposition 3.1. (*Cf.* [AlFaLaSo]) *Let γ be a linear symplectomorphism of $\mathbb{R}^{2n}$, and $2k$ the dimension of the fixed point space of γ. Then the twisted Hochschild homology of $\mathbb{W}_{2n}$ is given by*

$$HH_p^\gamma(\mathbb{W}_{2n}) = \begin{cases} \mathbb{K} & \text{for } p = 2k, \\ 0 & \text{for } p \neq 2k. \end{cases} \tag{3.3}$$

Proof. Although the proposition can be extracted from [NePfPoTa], let us give the (standard) argument. First note that there is a decomposition of the symplectic vector space $W = \mathbb{R}^{2n}$ as $W = W^\gamma \oplus W^\perp$, where $W^\gamma = \ker(1-\gamma)$ is the fixed point space of γ and $W^\perp = \mathrm{im}(1-\gamma)$ its symplectic orthogonal. Since $\gamma \in \mathrm{Sp}_{2n}$, this is a symplectic decomposition, and by assumption $\dim(W^\gamma) = 2k$. Choose a symplectic basis $(y_1, \cdots, y_{2k})$ of W^γ and extend it to a symplectic basis $(y_1, \cdots, y_{2n})$ of W. Observe now that the filtration by powers of $\hbar$ induces a spectral sequence with E^0-term the classical twisted Hochschild homology of the polynomial algebra $A_{2n} = \mathbb{K}[p_1, \cdots, p_n, q_1, \cdots, q_n]$. To compute this homology, one chooses a Koszul resolution of A:

$$0 \longleftarrow A \xleftarrow{m} A^e \xleftarrow{\partial} \ldots \xleftarrow{\partial} A^e \otimes \Lambda^{2n-1} W^* \xleftarrow{\partial} A^e \otimes \Lambda^{2n} W^* \longleftarrow 0,$$

where the differential ∂ is defined in the usual way by

$$\partial(a_1 \otimes a_2 \otimes dy_{i_1} \wedge \ldots \wedge dy_{i_p}) = \sum_{j=1}^{p} (-1)^j \left(y_{i_j} a_1 \otimes a_2 - a_1 \otimes y_{i_j} a_2\right) \otimes dy_{i_1} \wedge \ldots \wedge \widehat{dy_{i_j}} \wedge \ldots \wedge dy_{i_p}.$$

This is a projective resolution of A in the category of A-bimodules. Since by definition, cf. (3.1), $HH_\bullet^\gamma(A) = \mathrm{Tor}_\bullet^{A^e}(A_\gamma, A)$, we apply the functor $A_\gamma \otimes_{A^e} -$ to this resolution and compute the cohomology. This yields the complex

$$0 \longleftarrow A_\gamma \xleftarrow{m} A_\gamma \otimes W^* \xleftarrow{\partial} \ldots \xleftarrow{\partial} A_\gamma \otimes \Lambda^{2n-1} W^* \xleftarrow{\partial} A_\gamma \otimes \Lambda^{2n} W^* \longleftarrow 0$$

with differential

$$\partial(a \otimes dy_{i_1} \wedge \ldots \wedge dy_{i_p}) = \sum_{j=1}^{p} (-1)^j \left(y_{i_j} a - \gamma(y_{i_j}) a\right) \otimes dy_{i_1} \wedge \ldots \wedge \widehat{dy_{i_j}} \wedge \ldots \wedge dy_{i_p}.$$

Since $y_1, \cdots y_{2k}$ form a basis of V^γ, their contribution in the differential will vanish. The decomposition $W = W^\gamma \oplus W^\perp$ yields a decomposition of exterior products

$$\Lambda^l W^* = \bigoplus_{p+q=l} \Lambda^p (W^\gamma)^* \oplus \Lambda^q (W^\perp)^*,$$

and from the previous remark we see that the differential on the $\Lambda^p(W^\gamma)^*$-part is zero. In the q-direction one finds a direct sum of degree shifted Koszul complexes of the ring A associated to the regular sequence $(y_{k+1} - \gamma(y_{k+1}), \cdots, y_{2n} - \gamma(y_{2n}))$. Recall that a Koszul complex of a regular sequence $\mathbf{x} = (x_1, \cdots, x_n)$ in an algebra A has homology concentrated in degree 0 equal to A/I, where $I = (\mathbf{x})A$. In our case this ideal is exactly the vanishing ideal of W^γ, and one finds

$$HH_\bullet^\gamma(A) = \Omega^\bullet_{A_{2k}/\mathbb{K}},$$

where $A_{2k} = \mathbb{K}[y_1, \cdots, y_{2k}]$ and $\Omega^\bullet_{A_{2k}/\mathbb{K}}$ is the algebra of Kähler differentials. Notice that this argument proves a twisted analogue of the Hochschild–Kostant–Rosenberg theorem.

We therefore find $E^0_{p,q} = \Omega^{p+q}_{A_{2k}/\mathbb{K}}$, and the differential $d^0 : E^0_{p,q} \to E^0_{p,q-1}$ is the algebraic version of Brylinski's Poisson differential [Br] on the symplectic variety W^γ. Using the symplectic duality transform this complex is isomorphic to the algebraic de Rham complex of W^γ shifted by degree. Therefore one finds

$$E^1_{p,q} = \mathbb{K} \quad \text{iff } p + q = 2k.$$

The spectral sequence collapses at this point and the result follows. □

It is not difficult to check that the cocycle

$$c_{2k} = \sum_{\sigma \in \mathrm{S}_{2k}} \mathrm{sgn}(\sigma) 1 \otimes y_{\sigma(1)} \otimes \ldots \otimes y_{\sigma(2k)}$$

is a generator of $HH^\gamma_{2k}(\mathbb{W}_{2n})$. Notice that the cocycle only involves the basis elements on W^γ.

3.3. **The external product.** Let A and B be algebras equipped with automorphisms γ_A and γ_B. Since, as we have seen, the twisted Hochschild cohomology is nothing but the ordinary Hochschild cohomology with values in a twisted bimodule, the external product in Hochschild cohomology (see e.g. [We, Sec. 9.4.]) yields a map

$$\# : HH^p_{\gamma_A}(A) \otimes HH^q_{\gamma_B}(B) \to HH^{p+q}_{\gamma_A \otimes \gamma_B}(A \otimes B). \tag{3.4}$$

Let us describe its construction. Denote by $B_\bullet(A)$ and $B_\bullet(B)$ the bar resolution of A resp. B in the category of bimodules. Their tensor product $B_\bullet(A) \otimes B_\bullet(B)$ carries the structure of a bisimplicial vector space, which, by the Eilenberg–Zilber theorem is chain homotopy equivalent to its diagonal. But the diagonal $\mathrm{Diag}(B_\bullet(A) \otimes B_\bullet(B))$ is naturally isomorphic to the Bar complex of $A \otimes B$. Explicitly, the Eilenberg–Zilber theorem is induced by the so called Alexander–Whitney map, see [We, Sec.8.5.], which in our case gives rise to maps $f_{pq} : B_{p+q}(A \otimes B) \to B_p(A) \otimes B_q(B)$. Taking the Hom in the category of bimodules over $A \otimes B$ to $A_\gamma \otimes B_\gamma$, combined with the natural map

$$\mathrm{Hom}_{A^e}(B_p(A), A_{\gamma_A}) \otimes \mathrm{Hom}_{B^e}(B_p(B), B_{\gamma_B}) \to \mathrm{Hom}_{A^e \otimes B^e}(B_p(A \otimes B), A_{\gamma_A} \otimes B_{\gamma_B})$$

yields a map $C^p(A) \otimes C^q(B) \to C^{p+q}(A \otimes B)$ commuting with the twisted differentials, which induces (3.4). The explicit expression on the level of Hochschild cocycles can be read off from the Alexander–Whitney mapping. We will only need the following special case:

Assume $\gamma_A = 1$, and for notational simplicity put $\gamma := \gamma_B$. As one easily observes from the definition of the differential (3.2), a cocycle of degree 0 in the Hochschild cochain complex is nothing but a γ-twisted trace. Recall that a γ-twisted trace on a $\mathbb{K}$-algebra B with a fixed automorphism $\gamma \in \mathrm{Aut}(B)$ is a linear functional $\mathrm{tr}_\gamma : B \to \mathbb{K}$ such that

$$\mathrm{tr}_\gamma(b_1 b_2) = \mathrm{tr}_\gamma(\gamma(b_2) b_1). \tag{3.5}$$

Taking the external product with such a cocycle yields the following:

Lemma 3.2. *Let τ_k be a Hochschild cocycle on an algebra A of degree k. Then, for a γ-twisted trace tr_γ on a $\mathbb{K}$-algebra B with an automorphism γ, the formula*

$$\tau^\gamma_k\big((a_0 \otimes b_0) \otimes \ldots \otimes (a_k \otimes b_k)\big) := \tau_k(a_0 \otimes \ldots \otimes a_k)\ \mathrm{tr}_\gamma(b_0 \cdots b_k),$$

defines a γ-twisted Hochschild cocycle of degree k on $A \otimes B$.

Proof. Of course, one can prove this by checking that $\tau_k^\gamma = \tau_k \# \operatorname{tr}_\gamma$, a fact which can be read off from the precise form of the Alexander–Whitney map, and using that the map # passes to cohomology, as in (3.4). However it can also be done by a direct computation:

$$\begin{aligned}
&(b_\gamma \tau_k^\gamma)\big((a_0 \otimes b_0) \otimes \cdots \otimes (a_{k+1} \otimes b_{k+1})\big) = \\
&\quad = \sum_{i=0}^{k} (-1)^i \, \tau_k(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{k+1}) \operatorname{tr}_\gamma(b_0 \cdots b_{k+1}) \\
&\qquad + (-1)^{k+1} \, \tau_k(a_{k+1} a_0 \otimes \cdots \otimes a_k) \operatorname{tr}_\gamma(\gamma(b_{k+1}) b_0 \cdots b_k) \\
&\quad = (b\tau_k)(a_0 \otimes \cdots \otimes a_{k+1}) \operatorname{tr}_\gamma(b_0 \cdots b_{k+1}) = 0,
\end{aligned}$$

since τ_k is a Hochschild cocycle. Here, b denotes the ordinary (untwisted) Hochschild coboundary operator on the complex $C^k(A)$ and we have used the twisted trace property (3.5) of tr_γ. □

3.4. **The cocycle.** In this subsection we will construct a cocycle of degree $2k$ in the twisted Hochschild complex of $\mathbb{W}_{2n}$, where the twisting is induced by $\gamma \in \mathrm{Sp}_{2n}$. We use the notation from the proof of Prop. 3.1 and recall, in particular, the symplectic decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{(n-k)}$, where $\mathbb{C}^k = \ker(1-\gamma)$ and $\mathbb{C}^{(n-k)} = \operatorname{Im}(1-\gamma)$. By this, the Weyl algebra has form $\mathbb{W}_{2n} = \mathbb{W}_{2k} \otimes \mathbb{W}_{2(n-k)}$, and we can use the external product to construct the Hochschild cocycle. For notational simplicity, put $\mathbb{W}_{2n}^{\mathrm{T}} := \mathbb{W}_{2k}$ and $\mathbb{W}_{2n}^{\perp} := \mathbb{W}_{2(n-k)}$. The computation of the twisted Hochschild cohomology of $\mathbb{W}$, $\mathbb{W}^{\mathrm{T}}$ and $\mathbb{W}^{\perp}$ follows from Prop. 3.1. The twisted Hochschild cohomology of $\mathbb{W}_{2n}$ is one dimensional and concentrated in degree $2k$. Since the twisting induced by γ is trivial on $\mathbb{W}_{2n}^{\mathrm{T}}$, its (ordinary, i.e., untwisted) Hochschild cohomology is also one dimensional and in degree $2k$. Finally, $\mathbb{W}_{2n}^{\perp}$ has cohomology concentrated in degree 0 and equal to $\mathbb{K}$. Therefore, the only way to construct a nontrivial twisted Hochschild cocycle on $\mathbb{W}_{2n}$ is by taking the external product of an untwisted Hochschild cocycle of degree $2k$ on $\mathbb{W}_{2n}^{\mathrm{T}}$ with a twisted trace on $\mathbb{W}_{2n}^{\perp}$.

In [FeFeSh], a Hochschild cocycle of degree $2k$ on $\mathbb{W}_{2k}$ was constructed. Let us recall its definition. For $0 \le i \ne j \le 2k$, denote by α_{ij} the Poisson tensor on $\mathbb{C}^{2k}$ on the i'th and j'th slot of the tensor product $\mathbb{W}_{2k}^{\otimes(2k+1)}$:

$$\begin{aligned}
\alpha_{ij}(a_0 \otimes \cdots \otimes a_{2k}) = \frac{1}{2} \sum_{l=1}^{k} \Bigg(& a_0 \otimes \cdots \otimes \frac{\partial a_i}{\partial p_l} \otimes \cdots \otimes \frac{\partial a_j}{\partial q_l} \otimes \cdots \otimes a_{2k} \\
& - a_0 \otimes \cdots \otimes \frac{\partial a_i}{\partial q_l} \otimes \cdots \otimes \frac{\partial a_j}{\partial p_l} \otimes \cdots \otimes a_{2k} \Bigg).
\end{aligned}$$

The operator $\pi_{2k} \in \operatorname{End}(\mathbb{W}_{2k}^{\otimes(2k+1)})$ is defined as

$$\pi_{2k}(a_0 \otimes \cdots \otimes a_{2k}) = \sum_{\sigma \in S_{2k}} \operatorname{sgn}(\sigma) \, a_0 \otimes \frac{\partial a_1}{\partial y_{\sigma(1)}} \otimes \cdots \otimes \frac{\partial a_{2k}}{\partial y_{\sigma(2k)}}.$$

Let $\mu_{2k} : \mathbb{W}_{2k}^{\otimes 2k+1} \to \mathbb{K}$ be the operator $\mu_{2k}(a_0 \otimes \cdots \otimes a_{2k}) = a_0(0) \cdots a_{2k}(0)$, where $a_i(0)$ is the constant term of a_i. With these operators at hand, define

$$\tau_{2k}(a) = \mu_{2k} \int_{\Delta^{2k}} \prod_{0 \le i < j \le 2k} e^{\hbar(2u_i - 2u_j + 1)\alpha_{ij}} \pi_{2k}(a) du_1 \wedge \ldots \wedge du_{2k}, \tag{3.6}$$

where $a := a_0 \otimes a_1 \otimes \ldots \otimes a_{2k} \in \mathbb{W}_{2k}^{\otimes 2k+1}$, and Δ^{2k} is the standard simplex in $\mathbb{R}^{2k+1}$. As proved in [FeFeSh, Sec. 2], this defines a nontrivial Hochschild cocycle of degree $2k$. It is an explicit cocycle representative of the only non-vanishing cohomology class.

In the "transverse direction", i.e., on $\mathbb{W}_{2n}^{\perp}$, we need a twisted trace. Fortunately, such traces have been constructed in [Fe00]. For this, we choose a γ-invariant complex structure on $V^{\perp}$, identifying $V^{\perp} \cong \mathbb{C}^{n-k}$ so that $\gamma \in U(n-k)$. The inverse Caley transform

$$c(\gamma) = \frac{1-\gamma}{1+\gamma}$$

is an anti-hermitian matrix, i.e., $c(\gamma)^* = -c(\gamma)$. With this, define

$$\mathrm{tr}_\gamma(a) := \mu_{2(n-k)} \left(\det{}^{-1}(1-\gamma^{-1}) \exp\left(\hbar\, c(\gamma^{-1})^{ij} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} \right) a \right),$$

where $c(\gamma^{-1})^{ij}$ is the inverse matrix of $c(\gamma^{-1})$ and we sum over the repeated indices $i, j = 1, \ldots, n$. It is proved in [Fe00, Thm. 1.1], see also [FeSchTa, Lem. 7.3], that this functional is a γ-twisted trace density, i.e., satisfies equation (3.5). Clearly, $\mathrm{tr}_\gamma(1) = \det^{-1}(1-\gamma^{-1})$, so we immediately see from Proposition 3.1 that tr_γ is independent of the choice of a complex structure. This is also explicitly proved in [Fe00], but we view it as a "cohomological rigidity".

With the Hochschild cocycle τ_{2k} on $\mathbb{W}_{2k}$ and the twisted trace density τ_γ on $\mathbb{W}_{2(n-k)}$ we can now define the twisted Hochschild cocycle τ_{2k}^γ on $\mathbb{W}_{2n} = \mathbb{W}_{2k} \otimes \mathbb{W}_{2(n-k)}$ of degree $2k$ using the formula given in Lemma 3.2.

As we have seen in section 3.2, the Lie algebra $\mathfrak{sp}_{2n}$ acts on $\mathbb{W}_{2n}$ by derivations. Since $\gamma \in \mathrm{Sp}_{2n}$, it will act on $\mathfrak{sp}_{2n}$ by the adjoint action. In the following, we will be interested in the γ-fixed Lie subalgebra under this action:

$$\mathfrak{h} := \mathfrak{sp}_{2n}^\gamma \cong \mathfrak{sp}_{2k} \oplus \mathfrak{sp}_{2(n-k)}^\gamma.$$

The isomorphism follows from the decomposition $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$, which is a decomposition of representations of the cyclic group generated by γ, $\mathbb{C}^k$ being the isotypical summand of the trivial one. In general, $\mathfrak{h}$ is a semisimple Lie subalgebra of $\mathfrak{sp}_{2n}$. The following is the twisted analog of Theorem 2.2 in [FeFeSh] and lists the properties of the cocycle obtained by means of Lemma 3.2:

Proposition 3.3. *The cochain τ_{2k}^γ is a cocycle of degree $2k$ in the twisted, normalized Hochschild complex which has the following properties:*

i) The cochain τ_{2k}^γ is $\mathfrak{h}$-invariant which means that

$$\sum_{i=0}^{2k} \tau_{2k}^\gamma(a_0 \otimes \ldots \otimes [a, a_i] \otimes \ldots \otimes a_{2k}) = 0 \quad \textit{for all } a \in \mathfrak{h}.$$

ii) The relation $\tau_{2k}^\gamma(c_{2k}) = \det^{-1}(1-\gamma^{-1})$ holds true.

iii) For every $a \in \mathfrak{h}$ one has

$$\sum_{i=1}^{2k} (-1)^i \tau_{2k}^\gamma(a_0 \otimes \ldots \otimes a_{i-1} \otimes a \otimes a_i \otimes \ldots \otimes a_{2k-1}) = 0.$$

Proof. It follows from Lemma 3.2 that τ_{2k}^γ is a twisted Hochschild cocycle of degree $2k$. Since τ_{2k} is normalized on $\mathbb{W}_{2n}$, see [FeFeSh, Thm. 2.2], the twisted cocycle τ_{2k}^γ is normalized as well. For the first property, we write an element in $a \in \mathfrak{h}$ acting

on $\mathbb{W}_{2n}$ as $a = x \otimes 1 + 1 \otimes y$ with $x \in \mathfrak{sp}_{2k}$ and $y \in \mathfrak{sp}^{\gamma}_{2(n-k)}$. Consequently, we verify the equality in *i)* for x and y seperately. For $x \in \mathfrak{sp}_{2k}$ this is nothing but [FeFeSh, Thm. 2.2 *i)*]. For $y \in \mathfrak{sp}^{\gamma}_{2(n-k)}$, one has

$$\sum_{i=0}^{2k} \mathrm{tr}_{\gamma}(a_0 \cdots [y, a_i] \cdots a_{2k}) = 0,$$

because y commutes with γ. Property *ii)* follows at once from the fact that $\tau_{2k}(c_{2k}) = 1$, cf. [FeFeSh, Thm. 2.2 *ii)*]. Finally, *iii)* again splits into two parts for $a = x \otimes 1 + 1 \otimes y$ as above. The first part vanishes because of [FeFeSh, Thm. 2.2 *iii)*]. The second is zero since τ_{2k} is normalized. □

4. Construction of a twisted trace density

4.1. **Twisted Lie algebra cohomology.** Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras. Recall that for a $\mathfrak{g}$ module M, the Lie algebra cochain complex is given by $C^k(\mathfrak{g}; M) := \mathrm{Hom}(\Lambda^k \mathfrak{g}, M)$, with differential $\partial_{\mathrm{Lie}} : C^k(\mathfrak{g}; M) \to C^{k+1}(\mathfrak{g}; M)$ defined as

$$\begin{aligned}(\partial_{\mathrm{Lie}} f)(x_1 \wedge \ldots \wedge x_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} x_i \cdot f(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_{k+1}).\end{aligned}$$

This forms a complex, i.e., $\partial^2_{\mathrm{Lie}} = 0$, and its cohomology is the Lie algebra cohomology $H^{\bullet}(\mathfrak{g}; M)$. Likewise, the relative Lie algebra cochain complex $C^k(\mathfrak{g}, \mathfrak{h}; M) = \mathrm{Hom}(\Lambda^k(\mathfrak{g}/\mathfrak{h}), M)^{\mathfrak{h}}$ is the subcomplex of $C^{\bullet}(\mathfrak{g}; M)$ consisting of $\mathfrak{h}$-invariant cochains vanishing when any of the arguments is in $\mathfrak{h}$. Its cohomology $H^{\bullet}(\mathfrak{g}, \mathfrak{h}; M)$ is the relative Lie algebra cohomology with coefficients in M.

For an algebra A over $\mathbb{K}$, we denote by $\mathfrak{gl}_N(A)$ the Lie algebra of $N \times N$ matrices with entries in A. On equals footing, an A-bimodule M yields a $\mathfrak{gl}_N(A)$-module $\mathfrak{M}_N(M)$ of $N \times N$ matrices with entries in M and module structure given by the matrix commutator combined with the left and right A-module structure. For $M = A^*_{\gamma}$, we therefore obtain the Lie algebra complex $C^{\bullet}(\mathfrak{gl}_N(A); \mathfrak{M}_N(A^*_{\gamma}))$. This is the twisted Lie algebra cohomology complex, and we denote the differential by $\partial^{\gamma}_{\mathrm{Lie}}$. Consider now the map $\phi_N : C^{\bullet}(A, A^*_{\gamma}) \to C^{\bullet}(\mathfrak{gl}_N(A); \mathfrak{M}_N(A^*_{\gamma}))$ given by

$$\begin{aligned}&\phi_N(\tau)(M_1 \otimes a_1, \ldots, M_k \otimes a_k)(M_0 \otimes a_0) := \\ &\qquad \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \tau\left(a_0 \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)}\right) \mathrm{tr}\left(M_0 M_{\sigma(1)} \ldots M_{\sigma(k)}\right),\end{aligned}$$

where $\tau \in C^k(A, A^*_{\gamma})$, $M_0, M_1, \ldots M_k \in \mathfrak{gl}_N(\mathbb{K})$, $a_0, \ldots, a_k \in A$, and tr is the canonical trace on $\mathfrak{gl}_N(\mathbb{K})$. It is immediately clear by inspection of the differentials that this defines a morphism

$$\phi_N : (C^{\bullet}(A), b_{\gamma}) \to \left(C^{\bullet}(\mathfrak{gl}_N(A); \mathfrak{M}_N(A^*_{\gamma})), \partial^{\gamma}_{\mathrm{Lie}}\right)$$

of cochain complexes. Using this morphism we define the following $2k$-cocycle in the Lie algebra complex:

$$\Theta^{N,\gamma}_{2k} := \phi_N(\tau^{\gamma}_{2k}). \tag{4.1}$$

For $N = 1$ we simply write Θ^{γ}_{2k} for this cocycle.

4.2. **The construction.** We now come to the actual construction, which is in fact just a twisted version of the construction in [FeFeSh]. As is well-known, any deformation quantization of a symplectic manifold is isomorphic to a Fedosov deformation. On the symplectic manifold G_0 this implies that there is a resolution

$$0 \to \mathcal{A}^\hbar(G_0) \longrightarrow \Omega^0(G_0, \mathcal{W}_{2n}) \xrightarrow{D} \Omega^1(G_0, \mathcal{W}_{2n}) \xrightarrow{D} \dots$$

of the space of global sections of the sheaf $\mathcal{A}^\hbar$, where $\Omega^p(G_0, \mathcal{W})$ is the space of p-forms with values in the Weyl algebra bundle $\mathcal{W}_{2n}$, and D is a so-called Fedosov connection. Recall that $\mathcal{W}_{2n}$ is the bundle of algebras defined by

$$\mathcal{W}_{2n} = F^{2n}_{\mathrm{Sp}} \times_{\mathrm{Sp}_{2n}} \mathbb{W}_{2n}, \tag{4.2}$$

where F^{2n}_{Sp} denotes the bundle of symplectic frames on the tangent bundle TG_0. Combining the wedge product of forms with the algebra structure in the fibers of $\mathcal{W}$ turns $\Omega^\bullet(G_0, \mathcal{W})$ into a graded algebra, with product denoted by $\bullet$. By definition, a Fedosov connection is a connection on the Weyl algebra bundle which is a derivation with respect to this product, i.e.,

$$D(\alpha \bullet \beta) = (D\alpha) \bullet \beta + (-1)^{\deg(\alpha)} \alpha \bullet (D\beta).$$

The above resolution identifies $\mathcal{A}^\hbar(G_0)$ as the space of flat sections in $\Omega^0(G_0, \mathcal{W})$ compatible with its algebra structure. The Fedosov connection can be decomposed as

$$D = \nabla + [A, -], \tag{4.3}$$

where ∇ is a symplectic connection on TG_0, i.e., $\nabla\omega = 0$, $[-,-]$ is the commutator with respect to the product $\bullet$, and $A \in \Omega^1(G_0, \mathbb{W}_{2n})$. Since $D^2 = 0$, the quantity

$$\nabla A + \frac{1}{2}[A, A] = \Omega \tag{4.4}$$

must be central, i.e., is a $\mathbb{C}[[\hbar]]$-valued two form. Since G is a proper étale Lie groupoid, the Weyl algebra bundle $\mathcal{W}_{2n}$ is automatically a G-bundle, i.e., carries an action of G. We choose D to be an invariant Fedosov connection with respect to this action. Then this construction actually yields a resolution of $\mathcal{A}^\hbar$ in the category of G-sheaves. The associated symplectic connection ∇ and $\mathcal{W}_{2n}$-valued 1-form A are therefore G-invariant. Let us recall now that the pull-back by s (or equivalently by t) extends $\mathcal{W}_{2n}$ (resp. D, ∇, A) in a natural way to a bundle (resp. to connections resp. a 1-form) defined over G_1. For convenience, we will denote in the following the thus obtained objects by the same symbol as their restrictions to G_0.

Consider now a sector $\mathcal{O}$ of G, that is, a minimal G-invariant component of B_0. Using the natural embedding $\iota : \mathcal{O} \hookrightarrow G_1$, we can pull back the bundle of Weyl algebras on G_1 to $\iota^*\mathcal{W}_{2n}$. As a pull-back, this bundle inherits a natural Fedosov connection $\iota^* D = \iota^*\nabla + \iota^* A$ defined by

$$(\iota^* D)(\iota^*\alpha) = \iota^*(D\alpha), \tag{4.5}$$

with Weyl curvature

$$(\iota^*\nabla)(\iota^* A) + \frac{1}{2}[\iota^* A, \iota^* A] = \iota^*\Omega. \tag{4.6}$$

By definition (4.5), restriction to $\mathcal{O}$ maps flat sections of $\Omega^\bullet(G_0, \mathcal{W}_{2n})$ to flat sections over $\mathcal{O}$. Combined with the natural inclusion $\mathcal{A}^\hbar(G_0) \to \Omega^0(G_0, \mathcal{W}_{2n})$ above as flat sections with respect to the connection D, we obtain a natural morphism of sheaves $\iota^{-1}\mathcal{A}^\hbar \to \iota^{-1}\mathcal{W}_{2n}$ which we write on sections as $a \mapsto \iota^* a$. Indeed notice that the map $\iota^* : \iota^{-1}\mathcal{A}^\hbar(\mathcal{O}) \to \Omega^0(\mathcal{O}, \iota^*\mathcal{W}_{2n})$ thus defined, depends on the germ at $\mathcal{O}$ of a

section of $\mathcal{A}^\hbar$ on G_0, since its definition uses an embedding as a flat section of D on G_0 before restricting to $\mathcal{O}$. Applying this construction on every sector $\mathcal{O} \subset B_0$, one obtains a map $\iota^* : \iota^{-1}\mathcal{A}^\hbar(B_0) \to \Omega^0(B_0, \iota^*\mathcal{W}_{2n})$.

After these preparations, define

$$\psi_D(a) := \frac{(-1)^k}{(2k)!}\Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A)(\iota^*a). \tag{4.7}$$

Lemma 4.1. *ψ_D is a well-defined morphism of sheaves $\psi_D : \iota^{-1}\mathcal{A}^\hbar \to \Omega^{\mathrm{top}}_{B_0}[[\hbar]]$ which depends only on the Fedosov connection D.*

Proof. We have already showed that the germ of $\psi_D(a)$ at $x \in B_0$ depends only on the germ of a at x in G_0, i.e., the morphism is defined on $\iota^{-1}\mathcal{A}^\hbar$. Next, we have to show that it is independent of the choice of A, for a given Fedosov connection D. In general, the splitting (4.3) is unique up to a $\mathfrak{sp}_{2n}$-valued 1-form on G_0. Since the Fedosov connection D is G-invariant, the restriction to B_0 of the difference between two choices of A is given by an $\mathfrak{h}$-valued 1-form on B_0. Therefore, it follows from Proposition 3.3 *iii)* that ψ_D only depends on the choice of D. □

Proposition 4.2. *ψ_D defines a θ-twisted trace density on $\iota^{-1}\mathcal{A}^\hbar$.*

Proof. The proof is the same as in [FeFeSh], except for the twisting: Consider $\varphi : \iota^{-1}\mathcal{A}^\hbar \otimes \iota^{-1}\mathcal{A}^\hbar \to \Omega^{\mathrm{top}-1}_{B_0}[[\hbar]]$ defined by

$$\varphi(a \otimes b) := \frac{(-1)^k}{(2k)!}\Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A \wedge \iota^*a)(\iota^*b).$$

In the definition of ι^*a, we extended a to a flat section in $\Omega^0(G_0, \mathcal{W}_{2n})$. By (4.5), ι^*a therefore is flat with respect to ι^*D which means that $\iota^*\nabla(\iota^*a) + [\iota^*A, \iota^*a] = 0$. Now we compute

$$\begin{aligned}
d\varphi(a,b) &= (2k-1)\Theta^\gamma_{2k}((\iota^*\nabla)\iota^*A \wedge \ldots \wedge \iota^*A \wedge \iota^*a)(\iota^*b) \\
&\quad + \Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A \wedge \iota^*\nabla\iota^*a)(b) \\
&\quad + \Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A \wedge \iota^*a)(\iota^*\nabla\iota^*b) \\
&= -\frac{2k-1}{2}\Theta^\gamma_{2k}([\iota^*A, \iota^*A] \wedge \iota^*A \ldots \wedge \iota^*A \wedge \iota^*a)(\iota^*b) \\
&\quad - \Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A \wedge [\iota^*A, \iota^*a])(\iota^*b) \\
&\quad - \Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A \wedge \iota^*a)([\iota^*A, \iota^*b]) \\
&= \Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A)(\theta(\iota^*a)(\iota^*b) - (\iota^*b)(\iota^*a)) \\
&= \psi_D(\theta(a)b - ba),
\end{aligned}$$

where we have used equation (4.6). In the last step we have used the fact that Θ^γ_{2k} is a twisted Lie algebra cocycle, together with the fact that the connection ι^*A is G-invariant: $\theta(\iota^*A) = \iota^*A$. □

Applying Proposition 2.2 we obtain a trace on the deformed convolution algebra explicitly given by

$$\mathrm{Tr}(a) = \int_{B_0} \frac{1}{(2\pi i\hbar)^k}\psi_D(a) = \int_{B_0} \frac{1}{(2\pi i\hbar)^k}\Theta^\gamma_{2k}(\iota^*A \wedge \ldots \wedge \iota^*A)(\iota^*a). \tag{4.8}$$

Remark 4.3. In this formula and the remainder of this paper, k will be regarded as an integer-valued function on B_0 (resp. on $\tilde{X}$) which for every loop $g \in B_0$ gives half of the dimension of the fixed point space of the action of the isotropy group G_x on T_xG_0, where $x = s(g)$. Clearly, k is constant on each sector $\mathcal{O} \subset B_0$ by construction.

4.3. **Twisting by vector bundles.** To evaluate the image of the general index map (1.4), it is convenient to twist the construction of the density ψ_D by an orbifold vector bundle E. By definition, an orbifold vector bundle is a G-vector bundle on G_0. The Fedosov construction can be twisted by a vector bundle, see [Fe96], by tensoring the Weyl algebra bundle $\mathcal{W}_{2n}$ with $\mathrm{End}(E)$. Choosing a connection ∇_E on E, there is a flat Fedosov connection D_E on $\mathcal{W}_{2n} \otimes \mathrm{End}(E)$ which may be written as $D_E = \nabla \otimes 1 + 1 \otimes \nabla_E + [A_E, -]$, modifying (4.3). The space of flat sections is a formal deformation $\mathcal{A}_E^\hbar(G_0)$ of the algebra $\Gamma(G_0, \mathrm{End}(E))$ of smooth sections of the bundle $\mathrm{End}(E)$. In this section, we will construct traces on the crossed product $\mathcal{A}_E^\hbar \rtimes G$. Notice that since $\mathcal{A}_E^\hbar \rtimes G$ is Morita equivalent to $\mathcal{A}^\hbar \rtimes G$, its Hochschild and cyclic homology are isomorphic and they have the same number of independent traces.

Since G acts on the orbifold vector bundle, the restriction of E (or more precisely s^*E) to B_0 carries a canonical fiberwise action of the cyclic structure θ. Let V be the typical fiber of E, which is a representation space of the finite group Γ generated by $\theta = \gamma$. In the local model, we therefore switch from $\mathbb{W}_{2n}$ to the Weyl algebra with twisted coefficients $\mathbb{W}_{2n}^V := \mathbb{W}_{2n} \otimes \mathrm{End}(V)$. As in Section 4.1, the $\mathbb{W}_{2n}^V$-bimodule $\mathbb{W}_{2n,\gamma}^{V*}$, with the right action, in both components, twisted by γ, yields a natural $\mathfrak{gl}(\mathbb{W}_{2n}^V)$-module. Again, there is a natural morphism $\phi_V : C^\bullet(\mathbb{W}_{2n}^V, \mathbb{W}_{2n,\gamma}^{V*}) \to C^\bullet(\mathfrak{gl}(\mathbb{W}_{2n}^V); \mathbb{W}_{2n,\gamma}^{V*})$ given by

$$\phi_V(\tau)(M_1 \otimes a_1, \ldots, M_k \otimes a_k)(M_0 \otimes a_0) := \\ \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\tau\left(a_0 \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)}\right) \mathrm{tr}_V\left(\gamma M_0 M_{\sigma(1)} \ldots M_{\sigma(k)}\right),$$

where $\tau \in C^k(\mathbb{W}_{2n}^V, \mathbb{W}_{2n,\gamma}^{V*})$, $M_0, M_1, \ldots M_k \in \mathrm{End}(V)$, $a_0, \ldots, a_k \in \mathbb{W}_{2n}^V$, and tr_V is the canonical trace on $\mathrm{End}(V)$. The appearance of the γ-twisted trace $\mathrm{tr}_{V,\gamma}$ can be explained by factorizing this maps as the Morita equivalence

$$\mathrm{tr}_{V,\gamma} : C^\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n,\gamma}^*) \to C^\bullet(\mathbb{W}_{2n}^V, \mathbb{W}_{2n,\gamma}^{V*}),$$

combined with the natural morphism $C^\bullet(\mathbb{W}_{2n}^V, \mathbb{W}_{2n,\gamma}^{V*}) \to C^\bullet(\mathfrak{gl}(\mathbb{W}_{2n}^V); \mathbb{W}_{2n,\gamma}^{V*})$ of Section 4.1. For any Γ-representation V we therefore define

$$\Theta_{2k}^{V,\gamma} := \phi_V(\tau_{2k}^\gamma) \in C^{2k}(\mathfrak{gl}(\mathbb{W}_{2n}^V); \mathbb{W}_{2n,\gamma}^{V*}). \tag{4.9}$$

Continuing as in Section 4.2, the trace density $\psi_{D_E} : \iota^{-1}\mathcal{A}_E^\hbar \to \Omega_{B_0}^{\mathrm{top}}[[\hbar]]$ is defined by

$$\psi_{D_E}(a) = \Theta_{2k}^{V,\gamma}(\iota^* A_E \wedge \ldots \wedge \iota^* A_E)(\iota^* a).$$

The analogues of Lemma 4.1 and Proposition 4.2 are proved in exactly the same manner. Therefore the integral over B_0 of this density gives a trace Tr_E on $\mathcal{A}_E^\hbar \rtimes G$. Its virtues lie in the following proposition. Recall from (1.4) that the trace defined in equation (4.8) induces a map $\mathrm{Tr}_* : K_{\mathrm{orb}}^0(X) \to \mathbb{K}$.

Proposition 4.4. *Let E and F be orbifold vector bundles on X which are isomorphic outside a compact subset. Then $[E]-[F]\in K^0_{\mathrm{orb}}(X)$, and one has*

$$\mathrm{Tr}_*([E]-[F])=\int_{\tilde X}\frac{1}{(2\pi i\hbar)^k\,m}\big(\psi_{D_E}(1)-\psi_{D_F}(1)\big).$$

Hereby, $m:\tilde X\to\mathbb{N}$ is the locally constant function which coincides for every sector $\mathcal{O}\subset B_0$ with $m_{\mathcal{O}}$, the order of the isotropy group of the principal stratum of $\mathcal{O}/G\subset\tilde X$.

Remark 4.5. Note that both $\psi_{D_E}(1)$ and $\psi_{D_F}(1)$ are G-invariant, and therefore define differential forms on $\tilde X$. Moreover, since E and F are isomorphic outside a compact subset of X, $\psi_{D_E}(1)-\psi_{D_F}(1)$ is compactly supported on $\tilde X$.

Proof. First recall from Section 1.2 that for every orbifold vector bundle E, the section space $\Gamma_{\mathrm{c}}(E)$ is a projective $\mathcal{C}^\infty_{\mathrm{c}}(G)$-module. Hence, $\Gamma_{\mathrm{c}}(E)$ defines a projection e in the matrix algebra $\mathfrak{M}_N(\mathcal{C}^\infty_{\mathrm{c}}(G))$ of $\mathcal{C}^\infty_{\mathrm{c}}(G)$, for some large N. Observe that e is a projection with respect to the convolution product on $\mathcal{C}^\infty_{\mathrm{c}}(G)$, and generally is not a projection valued matrix function. According to [Fe96, Thm. 6.3.1], the projection e now has an extension to a "quantized" projection $\hat e$ in $\mathfrak{M}_N(\mathcal{A}^{((\hbar))}\rtimes G)$. By definition, $\mathrm{Tr}_*([E])$ equals $\mathrm{Tr}(\hat e)$.

In the following, we will express $\mathrm{Tr}(\hat e)$ in terms of an integral over the inertia orbifold. We will do the computation locally, and use a partition of unity later to glue the formulas. When restricted to a small open set U of the orbifold X, the groupoid representing U is Morita equivalent to a transformation groupoid $M\rtimes\Gamma\rightrightarrows M$, where M is symplectomorphic to an open set of $\mathbb{R}^{2n}$ with the standard symplectic form, and Γ is a finite group acting by linear symplectomorphisms on M. The restriction of e to $M\rtimes\Gamma$ can be computed explicitly.

Any Γ-vector bundle E on M can be embedded into a trivial Γ-vector bundle of high enough rank. Denote by E' the complement of E in this trivial Γ-vector bundle, and let e^Γ be the projection valued Γ-invariant function corresponding to the decomposition $E\oplus E'$. Let $\hat e^\Gamma$ be the quantization of e^Γ. By [ChDo, Thm. 3], one has $\psi_{D_E}(1)=\psi_{D_E+D_{E'}}(\hat e^\Gamma)$ for some Fedosov connection $D_{E'}$ on E'.

Now let Π denote the function $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\delta_\gamma$ on $M\rtimes\Gamma$, where δ_γ denotes the element $(1,\gamma)$ in $\mathcal{C}^\infty(M)\rtimes\Gamma$. Observe that Π is a projection with respect to the convolution product on $\mathcal{C}^\infty_{\mathrm{c}}(M)\rtimes\Gamma$. The subspace $M_\Pi:=\big(\mathcal{C}^\infty_{\mathrm{c}}(M)\rtimes\Gamma\big)\cdot\Pi$ generated by Π in $\mathcal{C}^\infty_{\mathrm{c}}(M)\rtimes\Gamma$ is a left $\mathcal{C}^\infty_{\mathrm{c}}(M)\rtimes\Gamma$-module and a right $\mathcal{C}^\infty_{\mathrm{c}}(M)^\Gamma$-module. Tensoring with M_Π thus defines a map from the projective modules on $\mathcal{C}^\infty_{\mathrm{c}}(M)^\Gamma$ to $\mathcal{C}^\infty_{\mathrm{c}}(M)\rtimes\Gamma$. Hence, it induces a map on the corresponding projectors. It is not hard to see that e^Γ is mapped to e, and e can be expressed by e^Γ as follows,

$$e=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}e^\Gamma\delta_\gamma.$$

The quantizations of e and e^Γ also satisfy this relation, hence

$$\hat e=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\hat e^\Gamma\delta_\gamma.$$

The "loop space" B_0 of the transformation groupoid $M\rtimes\Gamma$ is $\coprod_{\gamma\in\Gamma}M^\gamma$, where M^γ is the fixed point set of γ. Hence, over U, the trace $\mathrm{Tr}^U_*([E])$ is computed as

follows,

$$\begin{aligned}\mathrm{Tr}_*^U([E]) &= \int_{B_0} \frac{1}{(2\pi i\hbar)^k}\psi_D(\hat{e}) = \sum_{\gamma\in\Gamma}\int_{M^\gamma} \frac{1}{(2\pi i\hbar)^k}\psi_D(\hat{e})\\ &= \sum_{\gamma\in\Gamma}\int_{M^\gamma} \frac{1}{(2\pi i\hbar)^k\,|\Gamma|}\psi_D\big(\hat{e}^\Gamma\big)\\ &= \sum_{\langle\gamma\rangle}\int_{M^\gamma} \frac{1}{(2\pi i\hbar)^k\,|\,\mathrm{C}(\gamma)|}\psi_D\big(\hat{e}^\Gamma\big)\\ &= \sum_{\mathcal{O}}\int_{\mathcal{O}/\Gamma} \frac{1}{(2\pi i\hbar)^k\,m_{\mathcal{O}}}\psi_D\big(\hat{e}^\Gamma\big).\end{aligned}$$

Hereby, $\langle\gamma\rangle$ denotes the conjugacy class of $\gamma\in\Gamma$, $\mathrm{C}(\gamma)$ is the centralizer of γ, $\mathcal{O}\subset B_0$ runs through the sectors of $M\rtimes\Gamma$, and $\mathcal{O}/\Gamma\subset\tilde{U}$ is the quotient of $\mathcal{O}$ in the inertia (sub)orbifold $\tilde{U}\subset\tilde{X}$. Let us briefly justify the last equality in this formula. To this end recall first the definition of an orbifold integral from Section 1.3 and second that each sector $\mathcal{O}/\Gamma$ coincides with the quotient of some connected component of M^γ by the centralizer $\mathrm{C}(\gamma)$. Let x be an element of the open stratum of this connected component M^γ_α with respect to the stratification by orbit types. By definition, $m_{\mathcal{O}}$ is given by the order of the isotropy group $\mathrm{C}(\gamma)_x$. The orbit of $\mathrm{C}(\gamma)$ through x then has $|\,\mathrm{C}(\gamma)|/m_{\mathcal{O}}$ elements, and the integral $\int_{\mathcal{O}/\Gamma}\mu$ over an invariant form μ coincides with $\frac{m_{\mathcal{O}}}{|\,\mathrm{C}(\gamma)|}\int_{M^\gamma_\alpha}\mu$. This proves the claimed equality (cf. also [FeSchTa, Sec. 5]).

Using the fact that $\psi_{D_E}(1)=\psi_{D_E+D_{E'}}(\hat{e}^\Gamma)$, we thus obtain the following equality:

$$\mathrm{Tr}_*^U([E]) = \int_{\tilde{X}_U}\frac{1}{(2\pi i\hbar)^k\,m}\psi_{D_E}(1). \tag{4.10}$$

Finally, note that the constructions of the projections e, and e^Γ, and of the bimodule M_Π are local with respect to X. Therefore, one can glue together the local expressions (4.10) for the traces Tr_*^U by a partition of unity over X. This completes the proof. □

5. A local Riemann–Roch theorem for orbifolds

Our goal in this section is to use Chern-Weil theory in Lie algebra cohomology to express the form $\psi_D(1)$ constructed in Proposition 4.2 by characteristic classes.

Recall that in Section 4, we have defined $\psi_D(a)$ as $\Theta^{N,\gamma}_{2k}(\iota^*A\wedge\ldots\wedge\iota^*A)(\iota^*a)$, where

$$\Theta^{N,\gamma}_{2k}\in C^\bullet\big(\mathfrak{gl}_N(\mathbb{W}_{2n}),\mathfrak{gl}_N(\mathbb{K})\oplus\mathfrak{sp}_{2k}(\mathbb{K})\oplus\mathfrak{sp}^\gamma_{2n-2k}(\mathbb{K});\mathfrak{M}_N(\mathbb{W}^*_{2n,\gamma})\big)$$

is a cocycle. Consider now the morphism $\mathrm{ev}_1:\mathfrak{M}_N(\mathbb{W}^*_{2n,\gamma})\to\mathbb{K}$, which is the evaluation at the identity. Because of the nontrivial γ-action, this is not a morphism of $\mathfrak{gl}_N(\mathbb{W}_{2n})$-modules, but of $\mathfrak{gl}_N(\mathbb{W}^\gamma_{2n})$-modules. Consequently, we put

$$\begin{aligned}\mathfrak{g} :=& \mathfrak{gl}_N(\mathbb{W}^\gamma_{2n}),\\ \mathfrak{h} :=& \mathfrak{gl}_N(\mathbb{K})\oplus\mathfrak{sp}_{2k}(\mathbb{K})\oplus\mathfrak{sp}^\gamma_{2n-2k}(\mathbb{K}),\end{aligned} \tag{5.1}$$

and consider the Lie algebra cohomology cochain complex $C^{2k}(\mathfrak{g},\mathfrak{h};\mathbb{K})$. It is in this cochain complex that we explicitly identify the cocycle $\mathrm{ev}_1\,\Theta^{N,\gamma}_{2k}$. Notice that this

suffices to compute $\psi_D(1)$, since the restricted connection $\iota^* A$ is by assumption invariant under the action of the cyclic structure θ.

5.1. **Chern-Weil theory.** In the following, we use Chern-Weil theory of Lie algebras to determine the cohomology groups $H^p(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ for $p \leq 2k$.

First, recall the construction of Lie algebra Chern-Weil homomorphism. As above, let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra with an $\mathfrak{h}$-invariant projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$. Define the curvature $C \in \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h})$ of pr by $C(u \wedge v) := [\mathrm{pr}(u), \mathrm{pr}(v)] - \mathrm{pr}([u, v])$. Let $(S^\bullet \mathfrak{h}^*)^{\mathfrak{h}}$ be the algebra of $\mathfrak{h}$-invariant polynomials on $\mathfrak{h}$ graded by polynomial degree. Define the homomorphism $\chi : (S^\bullet \mathfrak{h}^*)^{\mathfrak{h}} \to C^{2\bullet}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ by

$$\chi(P)(v_1 \wedge \cdots \wedge v_{2q}) = \frac{1}{q!} \sum_{\substack{\sigma \in S_{2q}, \\ \sigma(2i-1) < \sigma(2i)}} (-1)^\sigma P\big(C(v_{\sigma(1)},\ v_{\sigma(2)}), \cdots, C(v_{\sigma(2q-1)},\ v_{\sigma(2q)})\big).$$

The right hand side of this equation defines a cocycle, and the induced map in cohomology $\chi : (S^\bullet \mathfrak{h}^*)^{\mathfrak{h}} \to H^{2\bullet}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$ is independent of the choice of the projection pr. This is the Chern–Weil homomorphism.

In our case, that means with $\mathfrak{g}$ and $\mathfrak{h}$ given by Equation (5.1), the projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ is defined by

$$\mathrm{pr}(M \otimes a) := \frac{1}{N} \mathrm{tr}(M) a_2 + M a_0,$$

where a_j is the component of a homogeneous of degree j in y. The essential point about the Chern–Weil homomorphism in this case is contained in the following result.

Proposition 5.1. *For $N \gg n$, the Chern-Weil homomorphism*

$$\chi : (S^q \mathfrak{h}^*)^{\mathfrak{h}} \to H^{2q}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$$

is an isomorphism for $q \leq 2k$.

Proof. The proof of this result goes along the same lines as the proof of Proposition 4.2 in [FeFeSh], bearing in mind that $\mathfrak{h}$ is semisimple. The only difference is that we need the following result on the cohomology $H^\bullet(\mathfrak{g}; S^q \mathfrak{g}^*)$ for $q > 0$ proved in Corollary A.5:

$$H^p(\mathfrak{gl}_N(\mathbb{W}^\gamma_{2n}); S^q \mathfrak{M}_N(\mathbb{W}^\gamma_{2n})) = \begin{cases} 0, & \text{for } j < 2k, \\ \mathbb{K}^l, & \text{for } j = 2k. \end{cases}$$

□

5.2. **The algebra $W^\gamma_{k,n-k,N}$.** By the isomorphism proved in Proposition 5.1, it follows that the cohomology class

$$[\mathrm{ev}_1(\Theta^{N,\gamma}_{2k})] \in H^{2k}(\mathfrak{g}, \mathfrak{h}; \mathbb{K})$$

corresponds, under the Chern–Weil isomorphism, to a unique element P^γ_{2k} in $(S^{2k}\mathfrak{h})^{*\mathfrak{h}}$. To find this polynomial, we restrict the Chern-Weil homomorphism to a Lie subalgebra $W^\gamma_{k,n-k,N} \subset \mathfrak{g}$.

Pick coordinates $(p_1, \ldots, p_k, q_1, \ldots q_k)$ of $\mathbb{R}^{2k}$ and $(z_{k+1}, \ldots z_{n-k})$ of $\mathbb{C}^{n-k}$ as in Section 3. Recall that $\gamma \in \mathrm{Sp}_{2n}$ acts trivially on the $p_i, q_i,\ 1 \leq i \leq k$. Moreover, we assume that the $z_j,\ k+1 \leq j \leq n$ are chosen to diagonalize γ. Let K be the polynomial algebra $K := \mathbb{K}[q_1, \cdots, q_k, z_{k+1}, \cdots, z_n]$. Introduce the Lie algebra

$W_{k,n-k,N} := \mathrm{Der}(K) \rtimes \mathfrak{gl}_N(K)$, the semidirect product of polynomial vector fields by $\mathfrak{gl}_N(K)$. This is the Lie subalgebra of $\mathfrak{gl}_N(\mathbb{W}_{2n})$ of elements of the form

$$\sum_i f_i p_i \otimes 1 + \sum_j g_j \bar{z}_j \otimes 1 + \sum_k h_k \otimes M_k, \quad \text{with } f_i, g_j, h_k \in K.$$

This Lie algebra has a natural γ-action, and $W^\gamma_{k,n-k,N}$ is defined to be the γ-invariant part of $W_{k,n-k,N}$. The Lie algebra $\mathfrak{h}_1 = W^\gamma_{k,n-k,N} \cap \mathfrak{h}$ is isomorphic to

$$\mathfrak{gl}_k \oplus \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_l} \oplus \mathfrak{gl}_N,$$

where $\mathfrak{gl}_{i_s}$ corresponds to the eigenspace of the γ-action on $\mathbb{C}^n$ with a given eigenvalue. Given this Lie subalgebra $\mathfrak{h}_1 \subset W^\gamma_{n,n-k,N}$, we now consider the Chern-Weil homomorphism

$$\chi : (S^q \mathfrak{h}_1^*)^{\mathfrak{h}_1} \to H^{2q}\big(W^\gamma_{k,n-k,N}, \mathfrak{h}_1; \mathbb{K}\big). \tag{5.2}$$

Proposition 5.2. *For $q \leq k$, the Chern-Weil homomorphism* (5.2) *is injective.*

Proof. By definition, $H^\bullet(W^\gamma_{k,n-k,N}, \mathfrak{h}_1; \mathbb{K})$ is the cohomology of the Cartan–Eilenberg cochain complex

$$\Big(\mathrm{Hom}\Big(\bigwedge\nolimits^\bullet (W^\gamma_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\Big)^{\mathfrak{h}_1}, \partial_{\mathrm{Lie}}\Big).$$

Write down a basis of $W_{k,n-k,N}$ as follows:

$$q^{\alpha_i} z^{\beta_i} p_i \otimes 1,\ q^{\alpha_j} z^{\beta_j} \bar{z}_j \otimes 1,\ q^{\alpha_{st}} z^{\beta_{st}} \otimes E_{st}, \tag{5.3}$$

where $q^{\alpha_i} z^{\beta_i} p_i$, $q^{\alpha_j} z^{\beta_j} \bar{z}_j$, and $q^{\alpha_{st}} z^{\beta_{st}}$ are polynomials. In the above formulas, $\alpha_i, \alpha_j, \alpha_{st}, \beta_i, \beta_j, \beta_{st}$ are multi-indices. If $\alpha_i = (\alpha_i^1, \cdots, \alpha_i^k)$, then $q^{\alpha_i} := q_1^{\alpha_i^1} \cdots q_k^{\alpha_i^k}$. If $\beta_j = (\beta_j^1, \cdots, \beta_j^{n-k})$, then $z^{\beta_j} := z_1^{\beta_j^1} \cdots z_{n-k}^{\beta_j^{n-k}}$. Finally, E_{st}, $1 \leq s, t \leq N$ denotes the elementary matrix with 1 at the (s,t)-position and 0 everywhere else. Since γ acts diagonally on this basis, the γ-invariant elements in (5.3) form a basis of $W^\gamma_{k,n-k,N}$.

Next, consider the $\mathfrak{h}_1$-action on $\mathrm{Hom}\big(\bigwedge^\bullet (W^\gamma_{k,n-k,N}), \mathbb{K}\big)$. Note that the following elements act diagonally on $W^\gamma_{k,n-k,N}$ and commute with each other:

$$\sigma_1 := \sum_{i=1}^k q_i p_i \otimes 1,\ 1 \leq i \leq k \quad \text{and} \quad \sigma_2 := \sum_{j=1}^{n-k} z_j \bar{z}_j \otimes 1,\ 1 \leq j \leq n-k.$$

Let us write down the formulas for the action of these elements.

(1) σ_1 action.
   (a) $[\sigma_1, q^{\alpha_{i'}} z^{\beta_{i'}} p_{i'} \otimes 1] = \big(\big(\sum_{l=1}^k \alpha_{i'}^l\big) - 1\big) q^{\alpha_i} z^{\beta_i} p_i \otimes 1$, where $\alpha_{i'}^l$ is the l-th component of $\alpha_{i'}$.
   (b) $[\sigma_1, q^{\alpha_j} z^{\beta_j} \bar{z}_j \otimes 1] = \big(\sum_{l=1}^k \alpha_j^l\big) q^{\alpha_j} z^{\beta_j} \bar{z}_j \otimes 1$, where α_j^l is the l-th component of α_j.
   (c) $[\sigma_1, q^{\alpha_{st}} z^{\beta_{st}} \otimes E_{st}] = \big(\sum_{l=1}^k \alpha_{st}^l\big) q^{\alpha_{st}} z^{\beta_{st}} \otimes E_{st}$, where α_{st}^l is the l-th component of α_{st}.

(2) σ_2 action.

(a) $[\sigma_2, q^{\alpha_{i'}} z^{\beta_{i'}} p_{i'} \otimes 1] = \big(\sum_{l=1}^{n-k} \beta_{i'}^l\big) q^{\alpha_{i'}} z^{\beta_{i'}} p_{i'} \otimes 1$, where $\beta_{i'}^l$ is the l-th component of $\beta_{i'}$.

(b) $[\sigma_2, q^{\alpha_{j'}} z^{\beta_{j'}} \bar{z}_{j'} \otimes 1] = \big(\sum_{l=1}^{n-k} \beta_{j'}^l - 1\big) q^{\alpha_{j'}} z^{\beta_{j'}} \bar{z}_{j'} \otimes 1$, where $\beta_{j'}^l$ is the l-th component of $\beta_{j'}$. Note that $\bar{z}_i$ is not γ-invariant, and therefore does not belong to $W^\gamma_{k,n-k,N}$. If $q^{\alpha_{j'}} z^{\beta_{j'}} \bar{z}_{j'} \otimes 1$ is γ-invariant, $\sum_{l=1}^{n-k} \beta_{j'}^l$ has to be greater than or equal to 1.

(c) $[\sigma_2, q^{\alpha_{st}} z^{\beta_{st}} \otimes E_{st}] = \big(\sum_{l=1}^{n-k} \beta_{st}^l\big) q^{\alpha_{st}} z^{\beta_{st}} \otimes E_{st}$, where β_{st}^l is the l-th component of α_{st}.

In the following, we denote by $|\alpha_i|$ the sum of the components of α_i, and similarly for $|\beta_j|$. Since σ_1, σ_2 are in $\mathfrak{h}_1$, we know that only those elements of

$$\mathrm{Hom}\Big(\bigwedge\nolimits^\bullet (W^\gamma_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\Big)$$

which have eigenvalue 0 will contribute to the $\mathfrak{h}_1$-relative cohomology. Note that σ_2 acts only with nonnegative eigenvalues. Therefore, we can reduce our considerations to the σ_2-invariant Lie subalgebra of $W^\gamma_{k,n-k,N}$. This Lie subalgebra will be denoted by $\tilde{W}^\gamma_{k,n-k,N}$ and has the following basis:

$$q^{\alpha_i} p_i \otimes 1,\ q^{\alpha_{jj'}} z_j \bar{z}_{j'} \otimes 1,\ q^{\alpha_{st}} \otimes E_{st}, \tag{5.4}$$

where z_j and $z_{j'}$ have the same eigenvalue for the γ-action, and where $1 \le i \le k$, $1 \le j, j' \le n-k$, and $1 \le s, t \le N$. By passing to the quotient $\tilde{W}^\gamma_{k,n-k,N}/\mathfrak{h}_1$, no elements of the form $q_{i'} p_i \otimes 1$, $z_j \bar{z}_{j'} \otimes 1$ or $1 \otimes E_{st}$ remain. To compute the relative cohomology $H^\bullet(W^\gamma_{k,n-k,N}, \mathfrak{h}_1)$, we consider the absolute cochain complex

$$\Big(\mathrm{Hom}\,\big(\bigwedge\nolimits^\bullet (\tilde{W}^\gamma_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\big)^{\mathfrak{h}_1}, \partial_{\mathrm{Lie}}\Big).$$

The Lie algebra $\tilde{W}^\gamma_{k,n-k,N}$ is closely related to the Lie algebra W_n considered in Theorem 2.2.4 of [Fu]. In particular, by the same arguments from invariant theory which show Lemma 1 in the proof of [Fu, Thm. 2.2.4] one concludes that $\big(\mathrm{Hom}\,\big(\bigwedge^\bullet (\tilde{W}^\gamma_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\big)^{\mathfrak{h}_1}, \partial_{\mathrm{Lie}}\big)$ is generated by even degree polynomials. Let us explain this in the following in some detail.

First, denote the dual basis of (5.4) for $\tilde{W}^\gamma_{k,n-k,N}$ as follows:

$$\begin{aligned} w_{\alpha_i} &= (q^{\alpha_i} p_i \otimes 1)^* \in \mathrm{Hom}(\tilde{W}^\gamma_{k,n-k,N}, \mathbb{K}), \\ w_{\alpha_{jj'}} &= (q^{\alpha_{jj'}} z_j \bar{z}_{j'} \otimes 1)^* \in \mathrm{Hom}(\tilde{W}^\gamma_{k,n-k,N}, \mathbb{K}) \\ w_{\alpha_{st}} &= (q^{\alpha_{st}} \otimes E_{st})^* \in \mathrm{Hom}(\tilde{W}^\gamma_{k,n-k,N}, \mathbb{K}). \end{aligned}$$

Moreover, if I is a finite set of indices α_i, denote by w_I the antisymmetric product

$$w_I = w_{\alpha^1_{i_1}} \wedge \cdots \wedge w_{\alpha^l_{i_l}},$$

where $\alpha^1_{i_1} < \cdots < \alpha^l_{i_l}$ are the elements of I ordered by lexicographic (or some other fixed) order on I. Likewise define w_J and w_S for each finite set of indices $\alpha_{jj'}$ resp. α_{st}. Then, every element $\psi \in \mathrm{Hom}(\bigwedge^\bullet (\tilde{W}^\gamma_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K})^{\mathfrak{h}_1}$ can be written as a linear combination

$$\psi = \sum_{I,J,S} \psi_{I,J,S}\, w_I \wedge w_J \wedge w_S,$$

where $\psi_{I,J,S} \in \mathbb{K}$, I runs through all finite sets of indices α_i, J through the finite sets of indices $\alpha_{jj'}$ with $|\alpha_{jj'}| \geq 1$ and S through all finite sets of indices α_{st} with $|\alpha_{st}| \geq 1$. Note that the restriction on the elements of the sets J and S comes from the fact that $z_j \bar{z}_{j'} \otimes 1$ and $1 \otimes E_{st}$ vanish in the quotient $W^{\gamma}_{k,n-k,N}/\mathfrak{h}_1$.

Since ψ is $\mathfrak{h}_1$-invariant and the σ_1-action is diagonal, the sum of the eigenvalues of the σ_1-action of each single component of $w_I \wedge w_J \wedge w_S$ has to vanish, in case $\psi_{I,J,S} \neq 0$. In other words, if $\psi_{I,J,S} \neq 0$, we have the following identity:

$$\sum_{\alpha_i \in I} (|\alpha_i| - 1) + \sum_{\alpha_{jj'} \in J} (|\alpha_{jj'}|) + \sum_{\alpha_{st} \in S} (|\alpha_{st}|) = 0. \tag{5.5}$$

Suppose now that in $w_I \wedge w_J \wedge w_S$ there are $m_{I,J,S}$ elements of the form $w_i = (p_i \otimes 1)^*$. Then Equation (5.5) entails

$$m_{I,J,S} = \sum_{\substack{\alpha_i > 0 \\ \alpha_i \in I}} (|\alpha_i| - 1) + \sum_{\substack{\alpha_{jj'} > 0 \\ \alpha_{jj'} \in J}} (|\alpha_{jj'}|) + \sum_{\substack{\alpha_{st} > 0 \\ \alpha_{st} \in S}} (|\alpha_{st}|). \tag{5.6}$$

Furthermore, by the same arguments which show Lemma 1 in the proof of [Fu, Thm. 2.2.4], one concludes from the $\mathfrak{gl}_k$-invariance of ψ, that the w_i's have to be paired with distinct terms of w_{α_k} or $w_{\alpha_{jj'}}$ or $w_{\alpha_{st}}$ such that $|\alpha_k| \geq 2$, $|\alpha_{jj'}|, |\alpha_{st}| \geq 1$. Thus, one has for $\psi_{I,J,S} \neq 0$ that $m_{I,J,S}$ is less than the total number of terms of w_{α_k} , $w_{\alpha_{jj'}}$, and $w_{\alpha_{st}}$ appearing in $w_I \wedge w_J \wedge w_S$ with $|\alpha_k|$, $|\alpha_{jj'}|$, $|\alpha_{st}| \geq 1$. Thus one has

$$m_{I,J,S} \leq \sum_{\substack{\alpha_i > 0 \\ \alpha_i \in I}} 1 + \sum_{\substack{\alpha_{jj'} > 0 \\ \alpha_{jj'} \in J}} 1 + \sum_{\substack{\alpha_{st} > 0 \\ \alpha_{st} \in S}} 1. \tag{5.7}$$

Equations (5.6) and (5.7) together show that w_{α_i}, $w_{\alpha_{jj'}}$, and $w_{\alpha_{st}}$ cannot appear in $w_I \wedge w_J \wedge w_S$ for nonvanishing $\psi_{I,J,S}$, if $|\alpha_i| \geq 3$ or $|\alpha_{jj'}| \geq 2$, or $|\alpha_{st}| \geq 2$. Hence, by Equation (5.6) again, we have that $m_{I,J,S}$ is equal to the number of terms w_{α_i}, $w_{\alpha_{jj'}}$ and $w_{\alpha_{st}}$ showing up in $w_I \wedge w_J \wedge w_S$ with $|\alpha_i| = 2$, $|\alpha_{jj'}| = 1$, and $|\alpha_{st}| = 1$. Therefore,

$$m_{I,J,S} = \sum_{|\alpha_i|=2} 1 + \sum_{|\alpha_{jj'}|=1} 1 + \sum_{|\alpha_{st}|=1} 1.$$

This shows that the degree of $w_I \wedge w_J \wedge w_S$ with $\psi_{I,J,S} \neq 0$ is equal to $2m_{I,J,S}$, which is even. The above arguments also show that $\operatorname{Hom}\big(\bigwedge^{\bullet}(\tilde{W}^{\gamma}_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\big)^{\mathfrak{h}_1}$ is generated by even degree polynomials. This implies that the differential on the cochain complex $\operatorname{Hom}\big(\bigwedge^{\bullet}(\tilde{W}^{\gamma}_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\big)^{\mathfrak{h}_1}$ degenerates. Therefore, in order to prove that χ is injective for $q \leq k$, it is enough to show that for every nonzero polynomial $P \in (S^q\mathfrak{h}_1)^{*\mathfrak{h}_1}$ the image

$$\chi(P) \in H^{2q}(W^{\gamma}_{k,n-k,N}, \mathfrak{h}_1; \mathbb{K}) = \operatorname{Hom}\big(\bigwedge^{2q}(\tilde{W}^{\gamma}_{k,n-k,N}/\mathfrak{h}_1), \mathbb{K}\big)^{\mathfrak{h}_1}$$

does not vanish on $\bigwedge^{2q}(\tilde{W}^{\gamma}_{k,n-k,N}/\mathfrak{h}_1)$. But this follows from a straightforward check. $\square$

5.3. **Calculation of** $\mathrm{ev}_1(\Theta_{2k}^{N,\gamma})$**.** By Proposition 5.1 and Proposition 5.2, one obtains the following commutative diagram.

$$\begin{array}{ccc} (S^k\mathfrak{h}^*)^{\mathfrak{h}} & \longrightarrow & (S^k\mathfrak{h}_1^*)^{\mathfrak{h}_1} \\ \chi\Big\downarrow & & \chi\Big\downarrow \\ H^{2k}(\mathfrak{g},\mathfrak{h};\mathbb{K}) & \longrightarrow & H^{2k}(W_{k,n-k,N}^{\gamma},\mathfrak{h}_1;\mathbb{K}) \end{array} \tag{5.8}$$

The left vertical arrow in (5.8) is the isomorphism χ proved from Proposition 5.1. The right vertical arrow has been constructed in Proposition 5.2 and has been proved to be injective. The two horizontal arrows are restriction maps. Since $\mathfrak{h}$ and $\mathfrak{h}_1$ have the same Cartan subalgebra $\mathfrak{a}$ spanned by

$$q_ip_i\otimes 1,\ 1\le i\le k,\quad z_j\bar{z}_j\otimes 1,\ k+1\le j\le n,\quad 1\otimes E_r,\ 1\le r\le N,$$

and since invariant polynomials are uniquely determined by their values on $\mathfrak{a}$, the upper triangle is injective. This implies that the lower horizontal map is also injective. Therefore, to determine a polynomial $P_k^{\gamma}\in(S^k\mathfrak{h}^*)^{\mathfrak{h}}$ such that $\chi(P_k^{\gamma})=\mathrm{ev}_1(\Theta_{2k}^{N,\gamma})$, one only needs to work with the restriction of $\mathrm{ev}_1(\Theta_{2k}^{N,\gamma})$ to $W_{k,n-k,N}^{\gamma}$.

Let $X=X_1\oplus X_2\oplus X_3\in\mathfrak{sp}_{2k}(\mathbb{K})\oplus\mathfrak{sp}_{2(n-k)}^{\gamma}(\mathbb{K})\oplus\mathfrak{gl}_N(\mathbb{K})=\mathfrak{h}$. Define $(\hat{A}_\hbar\,\mathrm{Ch}_\gamma\,\mathrm{Ch})_q\in(S^q\mathfrak{h}^*)^{\mathfrak{h}}$ to be the homogeneous terms of degree q in the Taylor expansion of

$$\Big(\hat{A}_\hbar\,\mathrm{Ch}_\gamma\,\mathrm{Ch}\Big)(X):=\hat{A}_\hbar(X_1)\,\mathrm{Ch}_\gamma(X_2)\,\mathrm{Ch}(X_3),$$

where $\hat{A}_\hbar(X_1)$ is $\det\big(\frac{\hbar X_1/2}{\sinh(\hbar X_1/2)}\big)^{\frac{1}{2}}$, and $\mathrm{Ch}_\gamma(X_2)$ is $\mathrm{tr}_\gamma(\exp_\star(X_2))$ of the star exponential of X_2, and $\mathrm{Ch}(X_3)$ is $\mathrm{tr}(\exp(X_3))$. Recall now that $\mathrm{ev}_1(\Theta_{2k}^{N,\gamma})$ is given by the formula

$$\mathrm{ev}_1\,\Theta_{2k}^{N,\gamma}(v_1,\cdots,v_{2k})=\Theta_{2k}^{N,\gamma}(v_1,\cdots,v_{2k})(1).$$

Theorem 5.3. *For $N\gg n$, the following identity holds in $H^{2k}(\mathfrak{g},\mathfrak{h};\mathbb{K})$:*

$$[\mathrm{ev}_1\,\Theta_{2k}^{N,\gamma}]=(-1)^k\chi((\hat{A}_\hbar\,\mathrm{Ch}\,\mathrm{Ch}_\gamma)_k).$$

Proof. Let us construct an invariant polynomial $P_k^{\gamma}\in(S^\bullet\mathfrak{h}^*)^{\mathfrak{h}}$ as follows. Since P_k^{γ} is required to be invariant under the adjoint action, it is determined by its value on the Cartan subalgebra $\mathfrak{a}$ which is spanned by $p_iq_i\otimes 1,\ 1\le i\le k$, $z_j\bar{z}_j\otimes 1,\ 1\le j\le n-k$, and $1\otimes E_l,\ 1\le l\le N$. Define P_k^{γ} in $(S^k\mathfrak{h}^*)^{\mathfrak{h}}$ to be the unique homogeneous polynomial whose restriction to $\mathfrak{a}$ is

$$\begin{aligned}P_k^{\gamma}(M_1\otimes a_1\otimes b_1,\cdots,M_k\otimes a_k\otimes b_k)&=\mathrm{tr}(M_1\cdots M_k)\cdot\\ \cdot\,\mu_k\Big(\int_{[0,1]^k}\prod_{1\le i\le j\le n}e^{\hbar\psi(u_i-u_j)\alpha_{ij}}(a_1\otimes\ldots\otimes a_k)du_1\cdots du_k\Big)&\,\mathrm{tr}_\gamma(b_1\star\cdots\star b_k),\end{aligned}$$

with μ_k is as in Section 3.4.

Remark 5.4. If we define $P_k(M_1\otimes a_1,\cdots,M_k\otimes a_k)$ simply as $P_k^{\gamma}(M_1\otimes a_1\otimes 1,\cdots,M_k\otimes a_k\otimes 1)$, the polynomial P_k is the same as the one in the proof of Theorem 4.1 in [FeFeSh]. Clearly, P_k^{γ} is the product of P_k and tr_γ.

In the following, we prove that $[\mathrm{ev}_1\,\Theta^{N,\gamma}_{2k}] = (-1)^k\chi(P^\gamma_k)$. By $\mathfrak{h}$-invariance, it is enough to show the equality on the Cartan subalgebra $\mathfrak{a}$. Let us define $u_{ij}, v_{ir}, w_{is} \in W^\gamma_{k,n-k,N}$ for $i,j = 1,\cdots,k$, $r = 1,\cdots,N$, and $s = k+1,\cdots,n$ as follows:

$$u_{ij} := \begin{cases} \frac{1}{2}q_i^2 p_i, & i = j, \\ q_i q_j p_j & i \neq j, \end{cases} \qquad v_{ir} := q_i \otimes E_r, \quad w_{is} := q_i z_s \bar{z}_s.$$

It is not difficult to check that

$$[p_i, u_{ij}]_\hbar = q_j p_j, \quad [p_i, v_{ir}]_\hbar = E_r, \quad [p_i, w_{is}]_\hbar = z_s\bar{z}_s.$$

Since $\mathrm{pr}(u_{ij}) = \mathrm{pr}(v_{ir}) = \mathrm{pr}(w_{is}) = 0$, one has

$$C(p_i, u_{ij}) = -p_i q_i, \quad C(p_i, v_{ir}) = -E_r, \quad \text{and} \quad C(p_i, w_{is}) = -z_s\bar{z}_s.$$

Let x_i be of the form u_{jk} with $j \geq k$ or v_{jr}, or w_{js}. Then it is straightforward to check that

$$\mathrm{ev}_1\,\Theta^{N,\gamma}_{2k}(p_1 \wedge x_1 \cdots \wedge p_k \wedge x_k) = (-1)^k P^\gamma_k\Big(\frac{x_1}{q_1}, \cdots, \frac{x_k}{q_k}\Big).$$

Note that in the definition of P^γ_k, we did not change the component of tr_γ, so the computation is the same as in the proof of [FeFeSh, Thm. 4.1].

Now, we explicitly evaluate the polynomial P^γ_k on the diagonal matrices in the Cartan algebra. Let $X = Y + Z$, where $Y := \sum \nu_i q_i p_i + \sum \sigma_r E_r$ and $Z := \sum \tau_s z_s \bar{z}_s$, with $\nu_i, \sigma_r, \tau_s \in \mathbb{K}$. Consider the generating function

$$S(X) := \sum_{m=1}^{\infty} \frac{1}{m!} P^\gamma_m(X, \cdots, X).$$

Then

$$\begin{aligned} S(X) &= \sum_{m\geq 0} \frac{1}{m!} P^\gamma_m(X, \cdots, X) \\ &= \sum_{m\geq 0} \frac{1}{m!} \sum_{0\leq l\leq m} \frac{m!}{l!(m-l)!} P_l(\underbrace{Y, \cdots, Y}_{l})\,\mathrm{tr}_\gamma(\underbrace{Z \star \cdots \star Z}_{m-l}) \\ &= \sum_{m\geq 0} \sum_{0\leq l\leq m} \frac{1}{l!(m-l)!} P_l(Y, \cdots, Y)\,\mathrm{tr}_\gamma(Z^{\star(m-l)}) \\ &= \sum_{l\geq 0} \frac{1}{l!} P_l(Y, \cdots, Y) \sum_{k=m-l\geq 0} \frac{1}{k!}\,\mathrm{tr}_\gamma(Z^{\star k}). \end{aligned}$$

According to [FeFeSh, Thm 4.1], the first term

$$\sum_{l\geq 0} \frac{1}{l!} P_l(Y, \cdots, Y)$$

in the above equation is equal to $(\hat{A}_\hbar\,\mathrm{Ch})(Y)$. For the second term, we can pull the sum into tr_γ and find $\mathrm{tr}_\gamma(\exp_\star(Z))$. Hence,

$$S(X) = (\hat{A}_\hbar\,\mathrm{Ch})(Y)\,\mathrm{Ch}_\gamma(Z).$$

Since P^γ_k is the degree k component of S, it is equal to $(\hat{A}_\hbar\,\mathrm{Ch}\,\mathrm{Ch}_\gamma)_k$. □

The same argument proves the twisted analogue for the cocycle $\Theta^{V,\gamma}_{2k}$ defined in (4.9):

Theorem 5.5. *For* $\dim V^\gamma \gg n$*, the following identity holds:*

$$[\mathrm{ev}_1\, \Theta_{2k}^{V,\gamma}] = (-1)^k \chi((\hat{A}_\hbar \,\mathrm{Ch}_V\, \mathrm{Ch}_\gamma)_k).$$

Remark 5.6. In Theorem 5.5, Ch_V is a twisted Chern character on $\mathfrak{gl}_V(\mathbb{K})^\gamma$ defined by $\mathrm{Ch}_V(X) := \mathrm{tr}(\gamma \exp(X))$ for $X \in \mathfrak{gl}_V(\mathbb{K})^\gamma$.

6. The algebraic index for orbifolds

In this section, we use the local Riemann-Roch formula in Theorem 5.3 to prove an algebraic index theorem on a symplectic orbifold and thus confirm a conjecture by [FeSchTa]. As an application of this algebraic index theorem, we provide in Section 6.2 an alternative proof of the Kawasaki index theorem for elliptic operators on orbifolds [Ka].

6.1. **The conjecture by Fedosov–Schulze–Tarkhanov.** Recall the set-up as described in Section 1. Consider a G-invariant formal deformation quantization $\mathcal{A}^\hbar$ of G_0 with characteristic class $\Omega \in H^2(X, \mathbb{K})$. Using the twisted trace density of Section 3, the trace $\mathrm{Tr} : \mathcal{A}^\hbar \rtimes G \to \mathbb{K}$ on the crossed product defined by Equation (4.8) gives rise to an index map by (1.4). The following theorem computes this map in terms of characteristic classes on the inertia orbifold $\tilde{X}$ of X.

Theorem 6.1. *Let E and F be orbifold vector bundles on X, which are isomorphic outside a compact subset. Then we have*

$$\mathrm{Tr}_*([E]-[F]) = \int_{\tilde{X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i})}{\det(1-\theta^{-1}\exp(-\frac{R^\perp}{2\pi i}))} \hat{A}\Big(\frac{R^T}{2\pi i}\Big) \exp\Big(-\frac{\iota^*\Omega}{2\pi i\hbar}\Big).$$

Proof. First, observe that by definition the left hand side computes the pairing between the cyclic cocycle of degree 0 given by the trace, and K-theory. Therefore, the left hand side only depends on the K-theory class of $[E]-[F] \in K^0_{\mathrm{orb}}(X)$ and will not change when we add a trivial bundle to E and F. Indeed, the right hand side is invariant under such changes as well, as for a trivial bundle we have $\mathrm{Ch}_\theta = \mathrm{Ch} = 1$. Consequently, we can assume without loss of generality that $\mathrm{rk}(E) = \mathrm{rk}(F) \gg n$, enabling the use of the local Riemann–Roch Theorem 5.3 of the previous section.

Let us first compute the local index density of Theorem 5.3 of the trivial vector bundle. In this local computation we put $\theta = \gamma$ as the twisting automorphism. Notice that since A is G-invariant, $\iota^* A$ defined in Section 4.2, Eq. (4.5) is also G-invariant. Therefore,

$$\psi_D(1) = \mathrm{ev}_1(\Theta_{2k}^{N,\gamma})(\iota^* A \wedge \cdots \wedge \iota^* A)$$

can be identified with the help of Theorem 5.3 as follows. Since the symplectic connection and $\iota^* A$ are γ-invariant, it follows that the curvature $\iota^*\tilde{R}$ is also γ-invariant. As in [FeFeSh, Sect. 4.7], we thus get

$$F(\iota^* A(\xi), \iota^* A(\eta)) = \iota^*\tilde{R}(\xi,\eta) - \iota^*\Omega(\xi,\eta)$$

for vector fields ξ and η on B_0. In the formula above, $\tilde{R} \in \Omega^2(G_0, \mathfrak{sp}_{2n}^\gamma)$ is the curvature of ∇, and we have used Equation (4.6) to arrive at this result. To apply Theorem 5.3, we split the curvature

$$\iota^*\tilde{R} = \iota^*\tilde{R}^t + \iota^*\tilde{R}^\perp$$

according to the decomposition $\mathfrak{sp}^\gamma_{2n} = \mathfrak{sp}_{2k} \oplus \mathfrak{sp}^\gamma_{2(n-k)}$. Substituting this into the formula for $\mathrm{ev}_1(\Theta^\gamma_{2k})$ of Theorem 5.3, we find

$$\begin{aligned}\frac{(-1)^k}{(2k)!}\,\mathrm{ev}_1\,\Theta^\gamma_{2k}(\iota^* A^{2k}) &= (-1)^k\,\mathrm{ev}_1\,\Theta^\gamma_{2k}(\iota^* A \wedge \cdots \wedge \iota^* A)\\ &= \frac{1}{n!} P^\gamma_k\big((\iota^*\tilde{R} - \iota^*\Omega)^k\big)\\ &= \big(\hat{A}_\hbar(\iota^*\tilde{R}^T)\,\mathrm{Ch}(-\iota^*\Omega)\,\mathrm{Ch}_\gamma(\iota^*\tilde{R}^\perp)\big)_k\\ &= \hbar^k\Big(\hat{A}(R^T)\,\mathrm{Ch}\Big(-\frac{1}{\hbar}\iota^*\Omega\Big)\,\mathrm{tr}_\gamma\Big(\exp_\star\Big(\frac{\iota^*\tilde{R}^\perp}{\hbar}\Big)\Big)\Big)_k.\end{aligned}$$

By the constructions in [Fe00, Sec. 5] and the computations in [Fe00, Thm. 4.1], one has $\mathrm{tr}_\gamma(\exp_\star(\frac{\iota^*\tilde{R}^\perp}{\hbar})) = (\det(1-\gamma^{-1}\exp(-R^\perp)))^{-1}$. Inserting this expression into the above formula, we obtain (modulo exact forms)

$$\psi_D(1) = \mathrm{ev}_1(\Theta^\gamma_{2k}) = \hbar^k \frac{\hat{A}(R^T)\,\mathrm{Ch}(-\frac{\iota^*\Omega}{\hbar})}{\det(1-\gamma^{-1}\exp(-R^\perp))}.$$

For a general (i.e., nontrivial) vector bundle E, we put V equal to the typical fiber of E and use Theorem 5.5 to find (modulo exact forms)

$$\psi_{D_E}(1) = \mathrm{ev}_1(\Theta^{V,\gamma}_{2k}) = \hbar^k \frac{\mathrm{Ch}_\gamma(R^E)}{\det(1-\gamma^{-1}\exp(-R^\perp))}\hat{A}\Big(R^T\Big)\,\mathrm{Ch}\Big(-\frac{\iota^*\Omega}{\hbar}\Big). \tag{6.1}$$

Applying Proposition 4.4, the result now follows. □

The preceeding proof shows even a stronger result, namely the following local version of the algebraic index theorem which we expect to have wider applications.

Theorem 6.2. *Let E be an orbifold vector bundle E on X, and, as before, denote by k the locally constant function on the inertia orbifold $\tilde{X}$ which over each sector $\mathcal{O}$ coincides with $\frac{1}{2}\dim\mathcal{O}$. Then the form*

$$\frac{1}{\hbar^k}\psi_{D_E}(1) - \frac{\mathrm{Ch}_\theta(R^E)}{\det(1-\theta^{-1}\exp(-R^\perp))}\hat{A}\Big(R^T\Big)\,\mathrm{Ch}\Big(-\frac{\iota^*\Omega}{\hbar}\Big),$$

on $\tilde{X}$ is exact.

Proof. Choose $N >> n$ sufficiently large and consider the vector bundle $E \oplus N_X$, where N_X denotes the trivial orbifold vector bundle of rank N. Next choose Fedosov connections D_E for E and D for the symplectic orbifold X. Recall that then $D_E \oplus ND$ is a Fedosov connection for $E \oplus N_X$. Now all the assumptions necessary for the local index formula Eq. 6.1 are satisfied for both the vector bundles $E \oplus N_X$ and N_X. Hence the forms

$$\frac{1}{\hbar^k}\psi_{D_E\oplus ND}(1) - \frac{N + \mathrm{Ch}_\theta(R^E)}{\det(1-\theta^{-1}\exp(-R^\perp))}\hat{A}\Big(R^T\Big)\,\mathrm{Ch}\Big(-\frac{\iota^*\Omega}{\hbar}\Big)$$

and

$$\frac{N}{\hbar^k}\psi_D(1) - \frac{N}{\det(1-\theta^{-1}\exp(-R^\perp))}\hat{A}\Big(R^T\Big)\,\mathrm{Ch}\Big(-\frac{\iota^*\Omega}{\hbar}\Big)$$

are both exact. Since $\Psi_{D_E}(1) + N\Psi_D(1) - \Psi_{D_E\oplus ND}(1)$ is exact as well, the claim then follows. □

6.2. **The Kawasaki index theorem.** In this section, we derive Kawasaki's index theorem [Ka] from Theorem 6.1 for orbifolds. Hereby, we apply the methods introduced by Nest–Tsygan in [NeTs96], where the relation between formal and analytic index formulas has been studied for compact riemannian manifolds.

Let X be a reduced compact riemannian orbifold, represented by a proper étale Lie groupoid $G_1 \rightrightarrows G_0$. We look at its cotangent bundle T^*X, which is a symplectic orbifold and is represented by the groupoid $T^*G_1 \rightrightarrows T^*G_0$ with the canonical (invariant) symplectic structure ω on T^*G_0. The standard asymptotic calculus of pseudodifferential operators on manifolds can be extended to X, as explained in Appendix B.

According to (B.2), the operator product on pseudodifferential operators defines an invariant star product $\star^{\mathrm{Op}}$ on T^*G_0. Since it is invariant, this star product descends to a star product $\star_X^{\mathrm{Op}}$ on T^*X. By (B.3) and (B.4), the operator trace on trace class operators defines a trace $\mathrm{Tr}^{\mathrm{Op}}$ on the deformation quantization of T^*G_0 and also a trace $\mathrm{Tr}_X^{\mathrm{Op}}$ on T^*X.

In the case of a reduced orbifold T^*X, one knows that the deformed groupoid algebra on T^*G constructed by Fedosov's approach as in [Ta] is Morita equivalent to the deformed G-invariant algebra on T^*G_0 with $\star^{\mathrm{Fe}}$ as product [NePfPoTa, Prop.6.5]. Recall that the latter is a deformation quantization for T^*X. By this Morita equivalence, Tr as defined by (4.8) reduces to a trace $\mathrm{Tr}_X^{\mathrm{Fe}}$ on the deformation quantization $\star_X^{\mathrm{Fe}}$ of T^*X.

By the classification of star products on a symplectic manifold by Nest and Tsygan [NeTs95], the deformation quantization on T^*G_0 constructed in Appendix B is equivalent to the G-invariant deformation quantization $\star^{\mathrm{Fe}}$ of T^*G_0 obtained by Fedosov quantization with Weyl curvature equal to $-\omega$. Since G is a proper étale Lie groupoid, one can choose the equivalence Φ between $\star^{\mathrm{Op}}$ and $\star^{\mathrm{Fe}}$ to be G-invariant. The pull-back of the trace $\mathrm{Tr}_X^{\mathrm{Fe}}$ by Φ thus defines a trace on $\star_X^{\mathrm{Op}}$. We denote this pull-back trace by $\mathrm{Tr}_X^{\mathrm{Fe}}$ as well.

Proposition 6.3. *The pull-back trace* $\mathrm{Tr}_X^{\mathrm{Fe}}$ *is equal to the trace* $\mathrm{Tr}_X^{\mathrm{Op}}$.

Proof. Consider a smooth function f on T^*X with support in an open subset $O \subset T^*X$ such that O is isomorphic to the orbit space of a transformation groupoid $\tilde{O} \rtimes \Gamma$, where $\tilde{O}$ is a Γ-invariant open subset of some finite dimensional symplectic vector space on which Γ acts by linear symplectomorphisms. When restricted to O, the traces $\mathrm{Tr}_O^{\mathrm{Fe}}$ and $\mathrm{Tr}_O^{\mathrm{Op}}$ both correspond to operator traces, but are defined by different polarizations of $\tilde{O}$. By construction, $\mathrm{Tr}_X^{\mathrm{Op}}$ is defined by a real polarization, while $\mathrm{Tr}_X^{\mathrm{Fe}}$ is defined by a complex polarization. The Hilbert spaces corresponding to these different polarizations are related by a Γ-invariant unitary operator. Hence, one concludes that $\mathrm{Tr}_O^{\mathrm{Fe}} = \mathrm{Tr}_O^{\mathrm{Op}}$, and therefore $\mathrm{Tr}_X^{\mathrm{Fe}}(f) = \mathrm{Tr}_X^{\mathrm{Op}}(f)$.

To check that $\mathrm{Tr}_X^{\mathrm{Fe}}(f) = \mathrm{Tr}_X^{\mathrm{Op}}(f)$ for any compactly supported smooth function f on T^*X one now uses an appropriate partition of unity to reduce the claim to the local case which just has been proved. The proposition follows. □

The symbol $\sigma(D)$ of an elliptic operator D on X defines an isomorphism between two orbifold vector bundles E and F on T^*X outside a compact subset. Similar to [Fe96, Eq. (4.2.2)], the index of D can be computed by $\mathrm{Tr}_X^{\mathrm{Op}}([E]-[F])$. By Proposition 6.3, we can use $\mathrm{Tr}_X^{\mathrm{Fe}}([E]-[F])$ to calculate this. By Theorem 6.1 we

have

$$\mathrm{Tr}_X^{\mathrm{Fe}}([E]-[F]) = \int_{\widetilde{T^*X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta(\frac{R^E}{2\pi i} - \frac{R^F}{2\pi i})}{\det(1-\theta^{-1}\exp(-\frac{R^\perp}{2\pi i}))} \hat{A}\Big(\frac{R^T}{2\pi i}\Big)\, \mathrm{Ch}\Big(-\frac{\iota^*\Omega}{2\pi i\hbar}\Big).$$

Now recall that in the construction of the star product $\star_X^{\mathrm{FE}}$, we have fixed $-\iota^*\Omega$ to $\iota^*\omega$. Since the canonical symplectic form ω is exact, one concludes that $\mathrm{Ch}(-\frac{1}{\hbar}\iota^*\Omega)$ is equal to 1 in this case. Hence we obtain the following theorem as an application of Theorem 6.1.

Theorem 6.4. [Ka] *Given an elliptic operator D on a reduced compact orbifold X, one has*

$$\mathrm{index}(D) = \int_{\widetilde{T^*X}} \frac{1}{m} \frac{\mathrm{Ch}_\theta\left(\frac{\sigma(D)}{2\pi i}\right)}{\det\left(1-\theta^{-1}\exp\left(-\frac{R^\perp}{2\pi i}\right)\right)} \hat{A}\Big(\frac{R^T}{2\pi i}\Big),$$

where $\sigma(D)$ is the symbol of D.

Appendix A. Twisted Hochschild and Lie algebra cohomology

In this section we consider the cohomology of the Lie algebra $\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma)$ for $N \gg 0$, where $\mathbb{W}_{2n}$ is the (formal) Weyl algebra over $\mathbb{R}^{2n}$ and Γ is a finite group which is assumed to act effectively by symplectomorphisms on $\mathbb{R}^{2n}$. More precisely, we will compute for $0 \leq p \leq n_\Gamma$, $N \gg n$, and $q \in \mathbb{N}^*$ the Lie algebra cohomology groups

$$H^p\big(\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)^*\big), \tag{A.1}$$

where n_Γ is a natural number depending on Γ and S^qM denotes for every vector space M the q-th symmetric power. Clearly, if M is a bimodule over an algebra A, then S^qM carries the structure of a $\mathfrak{gl}_N(A)$-module as follows:

$$[a, m_1 \vee \ldots \vee m_q] = \sum_{i=1}^q m_1 \vee \ldots \vee [a, m_i] \vee \ldots \vee m_q,$$

where $a \in A$, $m_1, \ldots, m_q \in M$ and $[a, m_i] = a\, m_i - m_i\, a$.

A.1. **$\mathbb{Z}_2$-graded Hochschild and Lie algebra cohomology.** For the computation of the above Lie algebra cohomology we will make use of the super or in other words $\mathbb{Z}_2$-graded versions of Hochschild and Lie algebra (co)homology. Let us briefly describe the construction of these super homology theories. To this end consider first a $\mathbb{Z}_2$-graded unital algebra $A = A_0 \oplus A_1$. For a homogeneous element $a \in A$ we then denote by $|a|$ its degree, that means the unique element $i \in \mathbb{Z}_2$ such that $a \in A_i$. Now there exist uniquely defined face maps $b_j : C_p(A) \to C_{p-1}(A)$, $0 \leq j \leq p$ which satisfy the following relations for homogeneous $a_0, a_1, \ldots, a_p \in A$:

$$b_j(a_0 \otimes \ldots \otimes a_p) = \\ = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_p, & \text{for } 0 \leq j < p, \\ (-1)^{|a_p|\,(|a_0|+\ldots+|a_{p-1}|)} a_p a_0 \otimes a_1 \otimes \ldots \otimes a_{p-1}, & \text{for } j = p. \end{cases} \tag{A.2}$$

Moreover, one has degeneracy maps $s_j : C_p(A) \to C_{p+1}(A)$, $0 \leq j \leq p$, and cyclic operators $t_p : C_p(A) \to C_p(A)$ defined as follows:

$$\begin{aligned} s_j(a_0 \otimes \ldots \otimes a_p) &= a_0 \otimes \ldots \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \ldots \otimes a_p, \\ t_p(a_0 \otimes \ldots \otimes a_p) &= (-1)^{|a_p|\,(|a_0|+\ldots+|a_{p-1}|)}(a_p \otimes a_0 \otimes \ldots \otimes a_{p-1}). \end{aligned} \tag{A.3}$$

It is straightforward to check that these data give rise to a cyclic object in the category of super vector spaces, hence to the Hochschild and cyclic homology of the super algebra A.

As a particular example consider the super algebra $\mathbb{C}[\varepsilon]$, where ϵ is of degree 1 and satisfies $\varepsilon^2 = 0$ (cf. [FeTs89, Sec. 3.1]). One proves immediately, that for each p, the chains $\omega_p^1 := 1 \otimes \varepsilon \otimes \ldots \otimes \varepsilon$ and $\omega_p^2 := \varepsilon \otimes \varepsilon \otimes \ldots \otimes \varepsilon$ are Hochschild cycles which generate the $\mathbb{Z}_2$-graded Hochschild homology $HH_p(\mathbb{C}[\varepsilon])$. Moreover, one checks that $B(\omega_p^1) = B(\omega_p^0) = 0$ and $B(\omega_p^2) = \omega_{p+1}^1$, where ω_p^0 denotes the cycle $1 \otimes \ldots \otimes 1$. Consider Connes' periodicity exact sequence

$$\longrightarrow HH_p(\mathbb{C}[\varepsilon]) \xrightarrow{I} HC_p(\mathbb{C}[\varepsilon]) \xrightarrow{S} HC_{p-2}(\mathbb{C}[\varepsilon]) \xrightarrow{B} HH_{p-1}(\mathbb{C}[\varepsilon]) \longrightarrow,$$

which also holds in the $\mathbb{Z}_2$-graded version. Then one checks by induction on p that $HC_p(\mathbb{C}[\varepsilon])$ is generated by $[\omega_p^0]$ and $[\omega_p^2]$ in case p is even, and by $[\omega_p^0]$, if p is odd. Thus, one has in particular

$$HC_p(\mathbb{C}[\varepsilon]) \cong HC_p(\mathbb{C}) \oplus \mathbb{C} \quad \text{for every } p \in \mathbb{N}. \tag{A.4}$$

Next consider a super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The super Lie algebra homology or in other words $\mathbb{Z}_2$-graded Lie algebra homology $H_\bullet(\mathfrak{g}, \mathbb{C})$ is then defined as the homology of the complex

$$\longrightarrow \Lambda^l \mathfrak{g} \xrightarrow{d} \Lambda^{l-1}\mathfrak{g} \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^1 \mathfrak{g} \xrightarrow{d} \Lambda^0 \mathfrak{g} = \mathbb{C},$$

where $\Lambda^l \mathfrak{g} := \bigoplus_{p+q=l} E^p \mathfrak{g}_0 \oplus S^q \mathfrak{g}_1$ means the super exterior product, and

$$d(\xi_1 \wedge \ldots \wedge \xi_p) = \sum_{1 \le i < j \le p} [\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_p.$$

Clearly, every super algebra A gives rise to a super Lie algebra $\mathfrak{gl}_N(A)$. Recall that one has the following relation between the (super) Lie algebra homology and the $\mathbb{Z}_2$-graded cyclic homology of A (cf. [Lo, Thm. 10.2.5]):

$$H_\bullet(\mathfrak{gl}_N(A); \mathbb{C}) \cong \Lambda_\bullet(HC[1](A)), \qquad N \gg 0, \tag{A.5}$$

where $\Lambda_\bullet$ denotes the functor which associates to a graded vector space its graded symmetric algebra.

A.2. **Computation of the Lie algebra cohomology.** Let us come back to our original goal, the computation of the Lie algebra homology groups in (A.1). To this end recall first the following result which has been proved in various forms in [AlFaLaSo, NePfPoTa, DoEt].

Proposition A.1. *Let Γ be a finite group which acts (from the right) by automorphisms on an algebra A over a field $\Bbbk$. Then the Hochschild (co)homology of the convolution algebra $A \rtimes \Gamma$ statisfies*

$$HH_\bullet(A \rtimes \Gamma) = H_\bullet(A, A \rtimes \Gamma)^\Gamma, \tag{A.6}$$

$$HH^\bullet(A \rtimes \Gamma) = H^\bullet(A, A \rtimes \Gamma)^\Gamma \tag{A.7}$$

Now let us consider again the (formal) Weyl algebra $\mathbb{W}_{2n}$ on $\mathbb{R}^{2n}$ (over the field $\mathbb{K} = \mathbb{C}((\hbar))$) and let Γ act by symplectomorphisms on $\mathbb{R}^{2n}$. Then note that $H_p\big(\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)\big)$ is dual to $H^p(\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)^*)$, the Lie algebra cohomology we are interested in. Hence it suffices to determine the

homology groups $H_p\big(\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)\big)$. Define the super algebra A as the $\mathbb{Z}_2$-graded tensor product

$$A = (\mathbb{W}_{2n} \rtimes \Gamma) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon].$$

By the above definition of super Lie algebra homology it is now clear that

$$H_p(A, \mathbb{K}) \cong \bigoplus_{p+q=k} H_p(\mathbb{W}_{2n} \rtimes \Gamma; S^q\mathbb{W}_{2n} \rtimes \Gamma).$$

Recall from Prop. 3.1 that for every $\gamma \in \Gamma$ the (twisted) Hochschild homology $H_\bullet^\gamma(\mathbb{W}_{2n}) = H_\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n\gamma})$ is concentrated in degree $2k_{\langle\gamma\rangle}$, where $2k_{\langle\gamma\rangle}$ is the dimension of the fixed point space of γ (which depends only on the conjugacy class $\langle\gamma\rangle$). Using the twisted Hochschild (co)homology of $\mathbb{W}_{2n}$ and Prop. A.1 one can derive the following result exactly as in [AlFaLaSo], where the case of the nonformal Weyl algebra has been considered.

Theorem A.2. (*cf.* [AlFaLaSo, Thm. 6.1]) *Let Γ be a finite group which acts effectively by symplectomorphisms on $\mathbb{R}^{2n}$ and consider its induced action (from the right) on the formal Weyl algebra over $\mathbb{R}^{2n}$. Denote for every $p \in \mathbb{N}$ by $l_p(\Gamma)$ the number of conjugacy classes of elements of Γ having a p-dimensional fixed point space. Then the following formula holds for the Hochschild (co)homology of the crossed product algebra $\mathbb{W}_{2n} \rtimes \Gamma$:*

$$\dim_{\mathbb{K}} H_p(\mathbb{W}_{2n} \rtimes \Gamma) = \dim_{\mathbb{K}} H^{2n-p}(\mathbb{W}_{2n} \rtimes \Gamma, \mathbb{W}_{2n} \rtimes \Gamma) = l_p(\Gamma). \tag{A.8}$$

Now choose for every element γ of Γ a Hochschild cycle α_γ the homology class of which generates $H_{2k_{\langle\gamma\rangle}}(\mathbb{W}_{2n}, \mathbb{W}_{2n\gamma})$, and denote by $\sqcup$ the exterior (shuffle) product. Then it is clear that

$$(C_{\bullet-2k_{\langle\gamma\rangle}}(\mathbb{K}), b, B) \xrightarrow[\sqcup\alpha_\gamma]{} (C_\bullet(\mathbb{W}_{2n}), b_\gamma, B) \tag{A.9}$$

is a morphism of mixed complexes. Moreover, by the properties of $\mathbb{W}_{2n}$ it is even a quasi-isomorphism for twisted Hochschild homology, hence a quasi-isomorphism of mixed complexes. Next consider the following composition of morphisms of mixed complexes

$$\begin{aligned} \bigoplus_{\langle\gamma\rangle\in\mathrm{Conj}(\Gamma)} \big(C_{\bullet-2k_{\langle\gamma\rangle}}(\mathbb{K}), b, B\big) &\xrightarrow{\alpha} \big(C_\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n} \rtimes \Gamma), b, B\big) \xrightarrow{\iota} \big(C_\bullet(\mathbb{W}_{2n} \rtimes \Gamma), b, B\big) \\ (c_{\langle\gamma\rangle})_{\langle\gamma\rangle\in\mathrm{Conj}(\Gamma)} &\mapsto (c_{\langle\gamma\rangle} \sqcup \alpha_\gamma)_{\gamma\in\Gamma}, \end{aligned} \tag{A.10}$$

where $\mathrm{Conj}(\Gamma)$ denotes the set of conjugacy classes of Γ, and where we have used the natural identification

$$\big(C_\bullet(\mathbb{W}_{2n}, \mathbb{W}_{2n} \rtimes \Gamma), b, B\big) \cong \bigoplus_{\gamma\in\Gamma} \big(C_\bullet(\mathbb{W}_{2n}), b_\gamma, B\big).$$

One now checks easily that the α_γ can be chosen in such a way that

$$\alpha_{\gamma\tilde{\gamma}\gamma^{-1}} = \gamma\, \alpha_{\tilde{\gamma}} \quad \text{for all } \gamma, \tilde{\gamma} \in \Gamma. \tag{A.11}$$

Under this assumption on the α_γ the composition $\iota \circ \alpha$ is a quasi-isomorphism of mixed complexes. Let us show this in some more detail. By the choice of the α_γ it is clear that the image of the morphism α is invariant under the action of Γ. Moreover, α is injective on homology by construction. Since the restriction of ι to the invariant part is a quasi-isomorphism in Hochschild homology by Prop. A.1, a

simple dimension counting argument then shows by Thm. A.2 that the composition $\iota\circ$ is a quasi-isomorphism in Hochschild homology as well. Hence $\iota\circ\alpha$ is also a quasi-isomorphism of mixed complexes. Thus, if $2k$ is minimal among the dimensions of fixed point spaces of the elements of Γ, one gets

$$HC_p(\mathbb{W}_{2n} \rtimes \Gamma) = \begin{cases} 0 & \text{for } 0 \leq p < 2k, \\ \mathbb{K}^{l_{2k}(\Gamma)} & \text{for } p = 2k. \end{cases} \tag{A.12}$$

By construction and the Kunneth-isomorphism, the Hochschild complex $C_\bullet(A)$ is quasi-isomorphic to the tensor product complex $(C_\bullet(A \rtimes \Gamma), b) \otimes C_\bullet(\mathbb{C}[\varepsilon])$. Hence, by the above considerations

$$\bigoplus_{\langle\gamma\rangle \in \mathrm{Conj}(\Gamma)} \big(C_{\bullet - k_{\langle\gamma\rangle}}(\mathbb{K}[\varepsilon]), b, B\big) \longrightarrow (C_\bullet(A), b, B)$$

is a quasi-isomorphism of mixed complexes as well, which implies that

$$HC_p(A) = \begin{cases} 0 & \text{for } 0 \leq p < 2k, \\ \mathbb{K}^{2l_{2k}(\Gamma)} & \text{for } p = 2k. \end{cases} \tag{A.13}$$

Inspecting formula (A.5) both for $\mathbb{W}_{2n}$ and A one now obtains the main result of this section.

Theorem A.3. *Let γ be a linear symplectomorphism of $\mathbb{R}^{2n}$ of finite order, Γ be the cyclic group generated by γ, and put $2k := \dim(\mathbb{R}^{2n})^\gamma$. Then for $N \gg n$,*

$$H_p(\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)) = \begin{cases} 0 & \textit{for } 0 \leq p < 2k, \\ \mathbb{K}^l & \textit{for } p = 2k, \end{cases} \tag{A.14}$$

where $l := l_{2k}(\Gamma)$ is the number of elements of Γ having a fixed point space of dimension $2k$.

Remark A.4. In the "untwisted case", i.e. for Γ the trivial group, these Lie algebra homology groups have been determined in [FeTs89].

By Eq. (A.5) one concludes easily that the Lie algebra (co)homology groups of $\mathfrak{gl}_N(\mathbb{W}_{2n} \rtimes \Gamma)$ with values in $S^q\mathfrak{M}_N(\mathbb{W}_{2n} \rtimes \Gamma)$ are Morita invariant. But one knows that $\mathbb{W}_{2n} \rtimes \Gamma$ is Morita equivalent to the invariant algebra $\mathbb{W}_{2n}^\Gamma$ (see [DoEt]). Thus the preceding theorem entails immediately

Corollary A.5. *Under the assumptions from above, one has for the invariant algebra $\mathbb{W}_{2n}^\gamma$,*

$$H_p(\mathfrak{gl}_N(\mathbb{W}_{2n}^\gamma); S^q\mathfrak{M}_N(\mathbb{W}_{2n}^\gamma)) = \begin{cases} 0 & \textit{for } 0 \leq p < 2k, \\ \mathbb{K}^l & \textit{for } p = 2k. \end{cases} \tag{A.15}$$

We can extend the above Corollary A.5 to the Weyl algebra with twisted coefficients $\mathbb{W}_{2n}^V := \mathbb{W}_{2n} \otimes \mathrm{End}(V)$, where V is a complex vector space γ acts on diagonally.

Note that $\mathbb{W}_{2n}^V$ is Morita equivalent to $\mathbb{W}_{2n}$ by the bimodule $\mathbb{W}_{2n} \otimes V$, and $\mathbb{W}_{2n}^V \rtimes \Gamma$ is Morita equivalent to $\mathbb{W}_{2n} \rtimes \Gamma$ by the bimodule $(\mathbb{W}_{2n} \otimes V) \rtimes \Gamma$. We have that

$$\begin{aligned} HH^\bullet(\mathbb{W}_{2n}^V \rtimes \Gamma) &= HH^\bullet(\mathbb{W}_{2n} \rtimes \Gamma), \\ HH_\bullet(\mathbb{W}_{2n}^V \rtimes \Gamma) &= HH_\bullet(\mathbb{W}_{2n} \rtimes \Gamma). \end{aligned}$$

Hence, by the same arguments as Corollary A.5, we have the following result for the Lie algebra homology of $\mathfrak{gl}_N((\mathbb{W}_{2n}^V)^\gamma)$.

Corollary A.6. *For* $(\mathbb{W}_{2n}^V)^\gamma$*, one has when* $N \gg n$*,*

$$H_p(\mathfrak{gl}_N((\mathbb{W}_{2n}^V)^\gamma); S^q\mathfrak{M}_N((\mathbb{W}_{2n}^V)^\gamma)) = \begin{cases} 0 & \text{for } 0 \leq p < 2k, \\ \mathbb{K}^l & \text{for } p = 2k. \end{cases} \tag{A.16}$$

Appendix B. Asymptotic pseudodifferential calculus

In the following we sketch the construction of the asymptotic calculus of pseudodifferential operators on a proper étale Lie groupoid. Hereby, we adapt the presentation in [NeTs96] to the groupoid case. For more details on the pseudodifferential calculus for proper étale groupoids, we refer the reader to Hu's thesis [Hu]. See also [Wi] for the complete symbol calculus on riemannian manifolds, and [Pf98] for its application to deformation quantization.

Let $G \rightrightarrows G_0$ be a proper étale Lie groupoid and fix a G-invariant riemannian metric on G_0. Then check that the canonical action of G on the diagonal $\Delta \subset G_0 \times G_0$ can be extended to a G-action on a neighborhood $W \subset G_0 \times G_0$ of Δ, since G is proper étale. Next construct a cut-off function $\chi : G_0 \times G_0 \to [0,1]$ with the following properties:

(1) $\operatorname{supp}\chi \subset W$, and the restriction $\chi_{|W}$ is invariant under the diagonal action of G on W.
(2) One has $\chi(x,y) = \chi(y,x)$ for all $x, y \in G_0$.
(3) On a neighborhood of the diagonal, one has $\chi \equiv 1$.
(4) For every $x \in G_0$, the set $Q_x = \{y \in G_0 \mid (x,y) \in \operatorname{supp}(\chi)\}$ is compact and geodesically convex with respect to the chosen metric.

For every open $U \subset G_0$ denote by $\mathrm{S}^m(U)$, $m \in \mathbb{Z}$, the space of symbols of order m on U, that means the space of smooth functions a on T^*U such that in each local coordinate system of U and each compact set K in the domain of the local coordinate system there is an estimate of the form

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|^2)^{\frac{m-|\beta|}{2}}, \quad x \in K,\, \xi \in T_x^*G_0,\, \alpha, \beta \in \mathbb{N}^n,$$

for some $C_{K,\alpha,\beta} > 0$. Clearly, the $\mathrm{S}^m(U)$ are the section spaces of a sheaf S^m on G_0. Moreover, each S^m is even a G-sheaf, i.e. carries a right G-action on the stalks. Finally, one obtains two further symbol sheaves by putting

$$\mathrm{S}^\infty := \bigcup_{m\in\mathbb{Z}} \mathrm{S}^m, \quad \mathrm{S}^{-\infty} := \bigcap_{m\in\mathbb{Z}} \mathrm{S}^m.$$

Similarly, one constructs the presheaves Ψ^m of pseudodifferential operators of order $m \in \mathbb{Z} \cup \{-\infty, \infty\}$ on G_0. Next let us recall the definition of the symbol map σ and its quasi-inverse, the quantization map Op. The symbol map associates to every operator $A \in \Psi^m(U)$ a symbol $a \in \mathrm{S}^m(U)$ by setting

$$a(x,\xi) := A\big(\chi(\cdot,x)e^{i\langle \xi, \mathrm{Exp}_x^{-1}(\cdot)\rangle}\big)(x),$$

where Exp_x^{-1} is the inverse map of the exponential map on Q_x. The quantization map is given by

$$\begin{aligned} &\mathrm{Op} : \mathrm{S}^m(U) \to \Psi^m(U) \subset \operatorname{Hom}\big(\mathcal{C}_{\mathrm{cpt}}^\infty(U), \mathcal{C}^\infty(U)\big), \\ &\big(\operatorname{Op}(a)f\big)(x) := \int_{T_x^*G_0}\int_{G_0} e^{i\langle \xi, \mathrm{Exp}_x^{-1}(y)\rangle}\chi(x,y)a(x,\xi)f(y)\,dy\,d\xi. \end{aligned}$$

The maps σ and Op are now quasi-inverse to each other in the sense that the induced morphisms $\overline{\sigma}$ and $\overline{\mathrm{Op}}$ between the quotient sheaves $\mathrm{S}^\infty / \mathrm{S}^{-\infty}$ and $\Psi^\infty / \Psi^{-\infty}$ are isomorphisms such that $\overline{\mathrm{Op}}^{-1} = \overline{\sigma}$.

By the space $\mathrm{AS}^m(U)$, $m \in \mathbb{Z}$ of *asymptotic symbols* over an open $U \subset G_0$ one understands the space of all $q \in \mathcal{C}^\infty(T^*U \times [0,\infty))$ such that for each $\hbar \in [0,\infty)$ the function $q(-,\hbar)$ is in $\mathrm{S}^m(U)$ and such that q has an asymptotic expansion of the form

$$q \sim \sum_{k\in\mathbb{N}} \hbar^k a_{m-k},$$

where each a_{m-k} is a symbol in $\mathrm{S}^{m-k}(U)$. More precisely, this means that one has for all $N \in \mathbb{N}$

$$\lim_{\hbar\searrow 0} \Big(q(-,\hbar) - \hbar^{-N} \sum_{k=0}^{N} \hbar^k a_{m-k} \Big) = 0 \quad \text{in } \mathrm{S}^{m-N}(U).$$

Clearly, the $\mathrm{AS}^m(U)$ are the sectional spaces of a G-sheaf AS^m. By forming the union resp. intersection of the sheaves AS^m like above one obtains two further G-sheaves AS^∞ and $\mathrm{AS}^{-\infty}$. Moreover, since G acts on these in a natural way, we also obtain the sheaf $\mathrm{AS}_G^{\pm\infty} := (\mathrm{AS}^{\pm\infty})^G$ of invariant asymptotic symbols and the convolution algebra $\mathrm{AS}^\infty \rtimes G$. For $m \in \mathbb{Z} \cup \{-\infty,\infty\}$ consider now the subsheaves $\mathrm{JS}^m \subset \mathrm{AS}^m$ and $\mathrm{JS}_G^m \subset \mathrm{AS}_G^m$ consisting of all (invariant) asymptotic symbols which vanish to infinite order on $\hbar = 0$. The quotient sheaves $\mathbb{A}^m := \mathrm{AS}^m / \mathrm{JS}^m$ and $\mathbb{A}_G^m := \mathrm{AS}_G^G / \mathrm{JS}_G^m$ can then be identified with the formal power series sheaves $\mathrm{S}^m[[\hbar]]$ resp. $\mathrm{S}_G^m[[\hbar]]$.

The operator product on Ψ^∞ now induces a (asymptotically associative) product on $\mathrm{AS}^\infty(G_0)$ by defining for $q, p \in \mathrm{AS}^\infty(G_0)$

$$q \star p(-,\hbar) := \begin{cases} \iota_{\hbar^{-1}} \sigma\big(\mathrm{Op}(\iota_\hbar q(-,\hbar)) \circ \mathrm{Op}(\iota_\hbar p(-,\hbar)) \big) & \text{if } \hbar > 0, \\ q(-,\hbar) \cdot p(-,\hbar) & \text{if } \hbar = 0. \end{cases} \tag{B.1}$$

Hereby, $\iota_\hbar : \mathrm{S}^\infty(G_0) \to \mathrm{S}^\infty(G_0)$ is the map which maps a symbol a to the symbol $(x,\xi) \mapsto a(x,\hbar\xi)$. By standard techniques of pseudodifferential calculus (cf. [Pf98]), one checks that the "star product" $\star$ has an asymptotic expansion of the following form:

$$q \star p \sim q \cdot p + \sum_{k=1}^{\infty} c_k(q,p)\, \hbar^k, \tag{B.2}$$

where the c_k are bidifferential operators on T^*G_0 such that

$$c_1(a,b) - c_1(b,a) = -i\{a,b\} \quad \text{for all symbols } a,b \in \mathrm{S}^\infty(G_0).$$

Hence, $\star$ is a star product on the quotient sheaf $\mathbb{A}^\infty$ and also on $\mathbb{A}_G^\infty$, since by construction the star product of invariant symbols is again invariant. Thus one obtains deformation quantizations for both the sheaf $\mathcal{A}_{T^*G_0}$ of smooth functions on TG_0 and the sheaf $\mathcal{A}_{T^*X} = \mathcal{A}_{T^*G_0}^G$ of smooth functions on the orbifold X represented by the groupoid G.

The invariant riemannian metric on G_0 gives rise to Hilbert spaces $L^2(G_0)$ and $L^2(X) = \pi_G L^2(G_0)$, where π_G is the orthogonal projection on the space of invariant functions. Hence there is a natural operator trace Tr_{L^2} on the space $\Psi_{\mathrm{cpt}}^{-\dim X}(G_0)$

of pseudodifferential operators of order $\leq -\dim X$ with compact support. Thus there is a map

$$\mathrm{Tr}^{\mathrm{Op}} : \mathbb{A}^{-\infty}_{\mathrm{cpt}}(G_0) \to \mathbb{C}[\hbar^{-1}, \hbar]], \quad q \mapsto \mathrm{Tr}_{L^2}\big(\mathrm{Op}\,\iota_\hbar(q(-,\hbar))\big),$$

which by construction has to be a trace with respect to $\star$ and is $\mathrm{ad}(\mathbb{A}^\infty)$-invariant. Moreover, by the global symbol calculus for pseudodifferential operators [Wi, Pf98] the following formula is satisfied as well:

$$\mathrm{Tr}^{\mathrm{Op}}(q) = \frac{1}{\hbar^{\dim X}} \int_{T^*G_0} q(-,\hbar)\,\omega^{\dim X}. \tag{B.3}$$

Finally, we obtain a trace $\mathrm{Tr}^{\mathrm{Op}}_G$ on the algebra $(\mathbb{A}^{-\infty}_{G,\mathrm{cpt}}(G_0), \star)$ of invariant asymptotic symbols as follows:

$$\mathrm{Tr}^{\mathrm{Op}}_G(q) := \mathrm{Tr}_{L^2}\big(\pi_G(\mathrm{Op}\,\iota_\hbar(q(-,\hbar)))\pi_G\big). \tag{B.4}$$

References

[AlFaLaSo] Alev, J., M.A. Farinati, T. Lambre and A.L.Solotar: *Homologie des invariants d'une algèbre de Weyl sous l'action d'une groupe fini*, J. of Algebra **232**, 564-577 (2000).

[AtSi] Atiyah, M.F. and I.M. Singer: *The index of elliptic operators. I*, Ann. of Math. (2) **87**, 484–530 (1986).

[AtSe] Atiyah, M.F. and G.B. Segal: *The index of elliptic operators. II*, Ann. of Math. (2) **87**, 531–545 (1986).

[BeGeVe] Berline, N., E. Getzler and M. Vergne: *Heat kernels and Dirac operators*, Springer-Verlag, Grundlehren Vol. **298** (1992).

[Br] Brylinski, J.L.: *A differential complex for Poisson manifolds*, J. Differential Geom. **28**, no. 1, 93–114 (1988).

[ChDo] Chen, P., and V. Dolgushev: *A Simple Algebraic Proof of the Algebraic Index Theorem*, Math. Res. Lett. **12**, no. 5-6, 655–671 (2005).

[Cr] Crainic, M.: *Cyclic cohomology of étale groupoids: the general case*, K-theory **17**, 319–362. (1999).

[DoEt] Dolgushev, V., and P. Etingof: *Hochschild cohomology of quantized symplectic orbifolds and the Chern–Ruan cohomology*, Int. Math. Res. Not. 2005, No. 27, 1657–1688 (2005).

[Fa] Farsi, C.: *K-theoretical index theorems for orbifolds*, Quart. J. Math. Oxford Ser. (2) **43**, no. 170, 183–200 (1992).

[Fe96] Fedosov, B.: *Deformation Quantization and Index Theory*, Akademie-Verlag, Berlin, 1996.

[Fe00] Fedosov, B.: *On G-trace and G-index in deformation quantization*, Conference Moshé Flato 1999 (Dijon), Lett. Math. Phys. **52** , no. 1, 29–49 (2002).

[Fe02] Fedosov, B.: *On the trace density in deformation quantization*, Halbout, Gilles (ed.), Deformation quantization, (Strasbourg, 2001), de Gruyter (Berlin), IRMA Lect. Math. Theor. Phys. 1, 67-83 (2002).

[FeSchTa] Fedosov, B., B.W. Schulze and N. Tarkhanov: *On index theorem for symplectic orbifolds*, Ann. Inst. Fourier (Grenoble) **54**, no. 5, 1601–1639 (2004).

[FeFeSh] Feigin, B:, G. Felder, and B. Shoikhet: *Hochschild cohomology of the Weyl algebra and traces in deformation quantization*, Duke Math. J. **127**, no. 3, 487–517 (2005).

[FeTs89] Feigin, B. and B. Tsygan: *Riemann-Roch theorem and Lie algebra cohomology*, in *I. Proceedings of the Winter School on Geometry and Physics* (Srn, 1988), Rend. Circ. Mat. Palermo (2) Suppl. No. 21, 15–52 (1989).

[FeTs87] Feigin, B. and B. Tsygan: *Additive K-theory*, in *K-theory, arithmetic and geometry*, (Moscow, 1984–1986), 67–209, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[Fu] Fuchs, D.: *Cohomology of Infinite dimensional Lie algebras*, Contemp. Soviet Math., Consultants Bureau, New York, 1986

[Hu] Hu, X.: *Transversally Elliptic Operators*, `arXiv:math.DG/0311069` (2003).

[Ka] Kawasaki, T.: *The index of elliptic operators over V-manifolds*, Nagoya Math. J. **84**, 135-157 (1981).

[Lo] Loday, J.L.: *Cyclic homology*, Springer Verlag, 1992.

[Mo] Moerdijk, I.: *Orbifolds as groupoids: an introduction*, Adem, A. (ed.) et al., Orbifolds in mathematics and physics (Madison, WI, 2001), Amer. Math. Soc., Contemp. Math. **310**, 205–222 (2002).

[MoPr] Moerdijk, I. and D. Pronk: *Orbifolds, sheaves and groupoids*, K-theory **12** (1997), 3–21.

[NeTs95] Nest, R., and B. Tsygan: *Algebraic index theorem*, Comm. Math. Phys **172**, 223–262 (1995).

[NeTs96] Nest, R., and B. Tsygan: *Formal versus analytic index theorems*, Intern. Math. Research Notes **11**, 557–564 (1996).

[NePfPoTa] Neumaier, N., M. Pflaum, H. Posthuma and X. Tang: *Homology of of formal deformations of proper étale Lie groupoids*, Journal f. die reine und angewandte Mathematik **593** (2006).

[Pf98] Pflaum, M.J.: *A deformation-theoretical approach to Weyl quantization on riemannnian manifolds*, Lett. Math. Physics **45** 277–294 (1998).

[Pf01] Pflaum, M.J.: *Analytic and geometric study of stratified spaces*, LNM **1768** Springer Verlag (Berlin), 2001.

[Pf03] Pflaum, M.J.: *On the deformation quantization of symplectic orbispaces*, Diff. Geom. Appl. **19** 343–368 (2003).

[Ta] Tang, X.: *Deformation quantization of pseudo Poisson groupoids*, to appear in Geom. Func. Analysis, `arXiv:math.QA/0405378` (2004).

[We] Weibel, Ch.: *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge Univ. Press 1995.

[Wi] Widom, H.: *A complete symbol for pseudodifferential operators*, Bull. Sci. Math. **104** (2), 19–63 (1980).

Markus J. Pflaum, `pflaum@math.uni-frankfurt.de`
Fachbereich Mathematik, Goethe-Universität Frankfurt/Main, Germany

Hessel Posthuma, `posthuma@maths.ox.ac.uk`
Mathematical Institute, University of Oxford, UK

Xiang Tang, `xtang@math.udavis.edu`
Department of Mathematics, University of California, Davis, USA