RELATIVE PAIRING IN CYCLIC COHOMOLOGY AND DIVISOR FLOWS

MATTHIAS LESCH, HENRI MOSCOVICI, AND MARKUS J. PFLAUM

ABSTRACT. We construct invariants of relative K-theory classes of multiparameter dependent pseudodifferential operators, which recover and generalize Melrose's divisor flow and its higher odd-dimensional versions of Lesch and Pflaum. These higher divisor flows are obtained by means of pairing the relative K-theory modulo the symbols with the cyclic cohomological characters of relative cycles constructed out of the regularized operator trace together with its symbolic boundary. Besides giving a clear and conceptual explanation to all the essential features of the divisor flows, this construction allows to uncover the previously unknown even-dimensional counterparts. Furthermore, it confers to the totality of these invariants a purely topological interpretation, that of implementing the classical Bott periodicity isomorphisms in a manner compatible with the suspension isomorphisms in both K-theory and in cyclic cohomology. We also give a precise formulation, in terms of a natural Clifford algebraic suspension, for the relationship between the higher divisor flows and the spectral flow.

INTRODUCTION

Cyclic cohomology of associative algebras, viewed as a noncommutative analogue of de Rham cohomology, together with its pairing with K-theory, was shown by CONNES [4] to provide a natural extension of the Chern-Weil construction of characteristic classes to the general framework of noncommutative geometry. In this capacity, cyclic cohomology has been successfully exploited to produce invariants for K-theory classes in a variety of interesting situations (see CONNES [5] for an impressive array of such applications, that include the proof of the Novikov conjecture in the case of Gromov's word-hyperbolic groups, cf. CONNES-MOSCOVICI [6]).

In this paper we present an application of this method to the construction of invariants of K-theory classes in the relative setting, which takes

---

2000 *Mathematics Subject Classification.* Primary 46L80; Secondary 58J30, 58J40.

The first named author was partially supported by Sonderforschungsbereich/Transregio 12 "Symmetries and Universality in Mesoscopic Systems" (Bochum–Duisburg/Essen–Köln–Warszawa).

The work of the second named author was partially supported by the US National Science Foundation award no. DMS-0245481.

The third named author was partially supported by the DFG.

advantage of the excision property in both topological K-theory and (periodic) cyclic cohomology (cf. WODZICKI [30], CUNTZ–QUILLEN [7]). Namely, we construct invariants of relative K-theory classes of multiparameter dependent pseudodifferential operators, which recover and generalize the divisor flow for suspended pseudodifferential operators introduced by MELROSE [21], as well as its multiparametric versions defined by LESCH–PFLAUM [19]. These invariants are obtained by pairing relative cyclic classes determined by the regularization à la Melrose of the operator trace and its symbolic boundary with the relative K-theory of the algebras of parametric pseudodifferential operators and of their symbols. By the very nature of the construction, the 'higher divisor flows' so obtained are homotopy invariant, additive and assume integral values. Besides providing a conceptual explanation for all their essential features, this interpretation leads naturally to the uncovering of the formerly 'missing' even dimensional higher eta invariants and of their associated higher divisor flows.

To outline the origins and content of this article in more precise terms, a modicum of notation will be necessary. Let M be a smooth compact Riemannian manifold without boundary, and let E be a hermitian vector bundle over M. We denote by $\mathrm{CL}^m(M,E)$ the classical (1-step polyhomogeneous) pseudodifferential operators of order m acting between the sections of E. It is well-known that the operator trace, which is defined on operators of order $m < -\dim M$, cannot be extended to a trace on the whole algebra $\mathrm{CL}^\infty(M,E) = \bigcup_{m\in\mathbb{R}} \mathrm{CL}^m(M,E)$. In fact, for M^n connected and $n > 1$, up to a scalar multiple there is only one tracial functional on $\mathrm{CL}^\infty(M,E)$, and that functional vanishes on pseudodifferential operators of order $m < -\dim M$, cf. WODZICKI [29]. This picture changes however if one passes to 'pseudodifferential suspensions' of the algebra $\mathrm{CL}^\infty(M,E)$. As shown by R.B. MELROSE [21], for a 'natural' pseudodifferential suspension $\Psi^\infty_{\mathrm{sus}}(M,E)$ of $\mathrm{CL}^\infty(M,E)$, the operator trace on operators of order $m < -\dim M - 1$ can be extended by a canonical regularization procedure to a trace on the full algebra. Melrose has used this regularized trace to 'lift' the spectrally defined η-invariant of ATIYAH-PATODI-SINGER [1] to an *eta* homomorphism $\eta : K_1^{\mathrm{alg}}(\Psi^\infty_{\mathrm{sus}}(M,E)) \to \mathbb{C}$, where K_1^{alg} stands for the algebraic K_1-theory group. Furthermore, by means of the variation of his generalized η-invariant, he defined the *divisor flow* between two invertibles of the algebra $\Psi^\infty_{\mathrm{sus}}(M,E)$ that are in the same component of the set of elliptic elements, and showed that it enjoys properties analogous to the spectral flow for self-adjoint elliptic operators.

Working with a slightly modified notion of pseudodifferential suspension, and for an arbitrary dimension $p \in \mathbb{N}$ of the parameter space, LESCH and PFLAUM [19] generalized Melrose's trace regularization to the p-fold suspended pseudodifferential algebra $\mathrm{CL}^\infty(M,E;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators. They also generalized Melrose's η-invariant to odd parametric dimensions, defining for $p = 2k+1$ the *higher*

η-invariant $\eta_{2k+1}(A)$ of an invertible $A \in \mathrm{CL}^\infty(M,E;\mathbb{R}^p)$. The appellative "eta" is justified by their result according to which any first-order invertible self-adjoint differential operator D can be canonically 'suspended' to an invertible parametric differential operator $\mathcal{D} \in \mathrm{CL}^1(M,E;\mathbb{R}^{2k+1})$, whose higher eta invariant $\eta_{2k+1}(\mathcal{D})$ coincides with the spectral η-invariant $\eta(D)$. On the negative side, in contrast with Melrose's η-homomorphism, these higher eta invariants are no longer additive on the multiplicative group of invertible elements. Nevertheless, the 'defect of additivity' is purely symbolic and hence local.

The starting point for the developments that make the object of the present paper was the fundamental observation that the higher η-invariants η_{2k+1}, when assembled together with symbolic corrections into *higher divisor flows* DF_{2k+1}, can be understood as the expression of the Connes *pairing* between the topological K-theory of the pair $\big(\mathrm{CL}^0(M,E;\mathbb{R}^{2k+1}), \mathrm{CL}^{-\infty}(M,E;\mathbb{R}^{2k+1})\big)$ and a certain canonical *relative cyclic cocycle*, determined by the regularized *graded trace* together with its symbolic coboundary. The first such invariant, for $k=0$, recovers Melrose's divisor flow, whose essential properties such as homotopy invariance, additivity and integrality, thus acquire a conceptual explanation. Of course, the same properties are shared by the higher divisor flows DF_{2k+1}. Furthermore, this framework allows us to find the appropriate even dimensional counterparts η_{2k}, $k>0$ as well as their corresponding higher divisor flows DF_{2k}. Moreover, Theorem 2.12 confers them a clear topological meaning, by showing that when taken collectively the higher divisor flows $\mathrm{DF}_\bullet$ implement the natural Bott isomorphisms between the topological $K_\bullet$-groups of the pair $\big(\mathrm{CL}^0(M,E;\mathbb{R}^\bullet), \mathrm{CL}^{-\infty}(M,E;\mathbb{R}^\bullet)\big)$ and $\mathbb{Z}$, in a manner compatible with the suspension isomorphisms in both K-theory and in cyclic cohomology.

As their basic properties indicate, and the very name given by Melrose is meant to suggest, the divisor flows are close relatives of the spectral flow. Our second main result gives a precise mathematical expression for this relationship, showing that spectral flow can be expressed as divisor flow via a natural Clifford algebraic 'suspension'. Because the standard complex Clifford representation comes in two 'flavors', which at the K-theoretical level provide the distinction between even and odd, the result is formulated accordingly, in two separate statements, Theorems 3.1 and 3.4.

Contents

1. Relative pairing in cyclic cohomology

1.1. **Relative cyclic cohomology.**

To establish the notation, we start by recalling in this section the definition of the relative homology and cohomology groups in terms of pairs. We then specialize this description to the case of cyclic cohomology.

Consider a short exact sequence of chain complexes (over the field $\mathbb{C}$)

$$0 \longrightarrow (K_\bullet, \partial_K) \stackrel{\kappa}{\longrightarrow} (A_\bullet, \partial_A) \stackrel{\alpha}{\longrightarrow} (B_\bullet, \partial_B) \longrightarrow 0, \tag{1.1}$$

where the differentials $\partial_A, \partial_B, \partial_K$ are of degree -1. Put

$$\widetilde{K}_k := \mathrm{Cone}(\alpha)_{k+1} := A_k \oplus B_{k+1}, \quad \widetilde{\partial} := \begin{pmatrix} \partial_A & 0 \\ -\alpha & -\partial_B \end{pmatrix} \tag{1.2}$$

and

$$\widehat{B}_k := \mathrm{Cone}(\kappa)_{k+1} := A_k \oplus K_{k-1}, \quad \widehat{\partial} := \begin{pmatrix} \partial_A & -\kappa \\ 0 & -\partial_K \end{pmatrix}. \tag{1.3}$$

In other words, $\widetilde{K}_\bullet$ is the mapping cone of α shifted by one degree, and $\widehat{B}_\bullet$ the mapping cone of κ.

Then $(\widetilde{K}_\bullet, \widetilde{\partial})$ and $(\widehat{B}_\bullet, \widehat{\partial})$ are chain complexes, and one finds the following natural chain maps:

$$\begin{array}{ll} K_\bullet \to \widetilde{K}_\bullet,\ c_k \mapsto (\kappa(c_k), 0), & \widehat{B}_\bullet \to B_\bullet,\ (a_k, b_{k-1}) \mapsto \alpha(a_k), \\ B_\bullet \stackrel{\iota}{\to} \widetilde{K}_{\bullet-1},\ b_k \mapsto (-1)^k (0, b_k), & A_\bullet \stackrel{\iota}{\to} \widehat{B}_\bullet,\ a_k \mapsto (a_k, 0), \\ \widetilde{K}_\bullet \stackrel{\pi}{\to} A_\bullet,\ (a_k, b_{k+1}) \mapsto a_k, & \widehat{B}_\bullet \stackrel{\pi}{\to} K_{\bullet-1},\ (a_k, c_{k-1}) \mapsto (-1)^k c_{k-1}, \end{array}$$

where we always assume that $a_l \in A_l$, $b_l \in B_l$ and $c_l \in K_l$. Moreover, $K_\bullet \to \widetilde{K}_\bullet$ and $\widehat{B}_\bullet \to B_\bullet$ are quasi-isomorphisms, and the connecting morphism in the long exact homology sequence for (1.1) becomes quite explicit. Namely, the long exact sequence reads

$$\longrightarrow H_k(\widetilde{K}_\bullet) \stackrel{\pi_*}{\longrightarrow} H_k(A_\bullet) \stackrel{\alpha_*}{\longrightarrow} H_k(B_\bullet) \stackrel{\iota_*}{\longrightarrow} H_{k-1}(\widetilde{K}_\bullet) \longrightarrow ,$$

resp.

$$\longrightarrow H_k(K_\bullet) \xrightarrow{\kappa_*} H_k(A_\bullet) \xrightarrow{\iota_*} H_k(\widehat{B}_\bullet) \xrightarrow{\pi_*} H_{k-1}(K_\bullet) \longrightarrow .$$

We shall say in this case that the complex $\widetilde{K}_\bullet$ (resp. $\widehat{B}_\bullet$) encodes the relative homology of $A_\bullet \to B_\bullet$ (resp. of $K_\bullet \to A_\bullet$).

In the dual situation one starts with a short exact sequence of cochain complexes

$$0 \longrightarrow (F^\bullet, d_F) \xrightarrow{\varepsilon} (E^\bullet, d_E) \xrightarrow{\delta} (Q^\bullet, d_Q) \longrightarrow 0, \tag{1.4}$$

where the differentials d_F, d_E, d_Q now have degree $+1$. Then the cochain complex $(\widetilde{Q}^\bullet, \widetilde{d})$, where

$$\widetilde{Q}^k := E^k \oplus F^{k+1}, \quad \widetilde{d} := \begin{pmatrix} d_E & -\varepsilon \\ 0 & -d_F \end{pmatrix} \tag{1.5}$$

is quasi-isomorphic to $(Q^\bullet, d_Q)$, and $(\widehat{F}^\bullet, \widehat{d})$ with

$$\widehat{F}^k := E^k \oplus Q^{k-1}, \quad \widehat{d} := \begin{pmatrix} d_E & 0 \\ -\delta & -d_Q \end{pmatrix}, \tag{1.6}$$

is quasi-isomorphic to $(F^\bullet, d_F)$. As above one obtains long exact cohomology sequences

$$\longrightarrow H^k(F^\bullet) \xrightarrow{\varepsilon_*} H^k(E^\bullet) \xrightarrow{\iota_*} H^k(\widetilde{Q}^\bullet) \xrightarrow{\pi_*} H^{k+1}(F^\bullet) \longrightarrow,$$

resp.

$$\longrightarrow H^k(\widehat{F}^\bullet) \xrightarrow{\pi_*} H^k(E^\bullet) \xrightarrow{\delta_*} H^k(Q^\bullet) \xrightarrow{\iota_*} H^{k+1}(\widehat{F}^\bullet) \longrightarrow .$$

Analogously to the homology case, we then say that the complex $\widetilde{Q}^\bullet$ (resp. $\widehat{F}^\bullet$) describes the relative cohomology of $F^\bullet \to E^\bullet$ (resp. of $E^\bullet \to Q^\bullet$).

Recall that a cochain complex $E^\bullet$ is said to be *dual* to the chain complex $A_\bullet$, if there is a non-degenerate bilinear pairing $\langle -, - \rangle : E^\bullet \times A_\bullet \to \mathbb{C}$ such that with respect to the pairing the differential d_E is adjoint to ∂_A. The main property of the complexes $\widetilde{K}_\bullet$ and $\widetilde{Q}^\bullet$ (resp. $\widehat{B}_\bullet$ and $\widehat{F}^\bullet$) now is that they behave nicely under duality pairings. This is expressed in the following proposition the proof of which is straightforward.

Proposition 1.1. *Assume that in the short exact sequences* (1.1) *and* (1.4) *the cochain complex $E^\bullet$ is dual to $A_\bullet$ and that $F^\bullet$ is dual to $B_\bullet$. Moreover, assume that ε is adjoint to α. Then with respect to the pairing*

$$\begin{aligned} &\langle -, - \rangle : \widetilde{Q}^\bullet \times \widetilde{K}_\bullet \to \mathbb{C}, \\ &\qquad ((\varphi_k, \psi_{k+1}), (a_k, b_{k+1})) \mapsto \langle \varphi_k, a_k \rangle + \langle \psi_{k+1}, b_{k+1} \rangle \end{aligned} \tag{1.7}$$

the complex $\widetilde{Q}^\bullet$ is dual to $\widetilde{K}_\bullet$, and this pairing induces a bilinear pairing $H^k(\widetilde{Q}^\bullet) \times H_k(\widetilde{K}_\bullet) \to \mathbb{C}$. Likewise, if $E^\bullet$ is dual to $A_\bullet$, $Q^\bullet$ dual to $K_\bullet$, and δ adjoint to κ, then $\widehat{F}^\bullet$ is dual to $\widehat{B}_\bullet$.

We now specialize the above notions to give a description of the relative cyclic homology and cohomology groups in the 'mapping cone setup'.

Recall that every unital $\mathbb{C}$-algebra $\mathcal{A}$ (possibly endowed with a locally convex topology) gives rise to a mixed complex $(C^\bullet(\mathcal{A}), b, B)$, where $C^\bullet(\mathcal{A})$ is the Hochschild cochain complex, b the Hochschild coboundary and B Connes' boundary (cf. CONNES [4], LODAY [20]). By definition, $C^k(\mathcal{A}) = \big(\mathcal{A} \otimes \mathcal{A}^{\otimes k}\big)^*$, and the operators b and B act on $\phi \in C^k(\mathcal{A})$ by

$$\begin{aligned} b\phi(a_0, \ldots, a_{k+1}) = \sum_{j=0}^{k} (-1)^j \, \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{k+1}) \\ + (-1)^{k+1} \phi(a_{k+1} a_0, a_1, \ldots, a_k), \end{aligned} \tag{1.8}$$

respectively

$$\begin{aligned} B\phi(a_0, \ldots, a_{k-1}) \; = \; \sum_{j=0}^{k-1} (-1)^{(k-1)j} \, \phi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) \\ - \sum_{j=0}^{k-1} (-1)^{(k-1)j} \, \phi(a_j, 1, a_{j+1}, \ldots, a_k, a_0, \ldots, a_{j-1}). \end{aligned} \tag{1.9}$$

Consider now the double complexes $\mathcal{B}C^{\bullet,\bullet}(\mathcal{A})$ and $\mathcal{B}C^{\bullet,\bullet}_{\mathrm{per}}(\mathcal{A})$. They consist of the non-vanishing components $\mathcal{B}C^{p,q}(\mathcal{A}) = C^{q-p}(\mathcal{A})$ for $q \geq p \geq 0$ resp. $\mathcal{B}C^{p,q}_{\mathrm{per}}(\mathcal{A}) = C^{q-p}(\mathcal{A})$ for $q \geq p$, and have B as horizontal and b as vertical differential. The *cyclic* resp. *periodic cyclic cohomology* groups of $\mathcal{A}$ are then given as follows:

$$HC^\bullet(\mathcal{A}) = H^\bullet(\mathrm{Tot}^\bullet_{\bigoplus} \mathcal{B}C^{\bullet,\bullet}(\mathcal{A})) \;\text{ and }\; HP^\bullet(\mathcal{A}) = H^\bullet(\mathrm{Tot}^\bullet_{\bigoplus} \mathcal{B}C^{\bullet,\bullet}_{\mathrm{per}}(\mathcal{A})).$$

In both cases the differential on the total complex is $b + B$. A (continuous) surjective homomorphism of algebras $\sigma : \mathcal{A} \to \mathcal{B}$ now induces a morphism of mixed complexes $\sigma^* : C^\bullet(\mathcal{B}) \to C^\bullet(\mathcal{A})$. Thus, we are in a 'relative situation' as in (1.5) and can express the corresponding *relative cyclic* resp. *periodic cyclic cohomology* accordingly. More precisely, the relative cyclic cohomology coincides with the cohomology of the total complex

$$\big(\,\mathrm{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}) \oplus \mathrm{Tot}^{\bullet+1}_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{B}), \widetilde{b+B}\,\big),$$

where the differential is given by

$$\widetilde{b+B} = \begin{pmatrix} b+B & -\sigma^* \\ 0 & -(b+B) \end{pmatrix}.$$

Explicitly, $\mathrm{Tot}^k_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}) \oplus \mathrm{Tot}^{k+1}_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{B}) \cong$

$$\cong \bigoplus_{p+q=k} \mathcal{B}C^{p,q}(\mathcal{A}) \oplus \mathcal{B}C^{p,q+1}(\mathcal{B}) = \mathrm{Tot}^k_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}, \mathcal{B}),$$

where $\mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B})$ is the double complex associated to the relative mixed complex $(C^{\bullet}(\mathcal{A},\mathcal{B}),\widetilde{b},\widetilde{B})$, which is given by $C^k(\mathcal{A},\mathcal{B})=C^k(\mathcal{A})\oplus C^{k+1}(\mathcal{B})$,

$$\widetilde{b}=\begin{pmatrix} b & -\sigma^* \\ 0 & -b \end{pmatrix}, \text{ and } \widetilde{B}=\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

Hence the relative cyclic cohomology $HC^{\bullet}(\mathcal{A},\mathcal{B})$, resp. the relative periodic cyclic cohomology $HP^{\bullet}(\mathcal{A},\mathcal{B})$ can be identified canonically with the cohomology of

$$\big(\operatorname{Tot}^{\bullet}_{\oplus}\mathcal{B}C^{\bullet,\bullet}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big) \quad \text{resp.} \quad \big(\operatorname{Tot}^{\bullet}_{\oplus}\mathcal{B}C^{\bullet;\bullet}_{\mathrm{per}}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big).$$

Of course, the "staircase trick" also works for the relative cyclic complex. As a consequence, each class in $HC^k(\mathcal{A},\mathcal{B})$ has a representative $(\varphi,\psi)\in C^k_\lambda(\mathcal{A})\oplus C^{k+1}_\lambda(\mathcal{B})$ with $b\varphi=\sigma^*\psi$, where $C^{\bullet}_\lambda$ stands for the subcomplex of cyclic cochains [4].

The preceding considerations can be dualized in an obvious fashion. Thus, $HC_{\bullet}(\mathcal{A},\mathcal{B})$ is the homology of $\big(\operatorname{Tot}^{\oplus}_{\bullet}\mathcal{B}C_{\bullet,\bullet}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big)$, where $\mathcal{B}C_{p,q}(\mathcal{A},\mathcal{B})=\mathcal{B}C_{p,q}(\mathcal{A})\oplus\mathcal{B}C_{p,q+1}(\mathcal{B})$,

$$\widetilde{b}=\begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \text{ and } \widetilde{B}=\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

Likewise, the *periodic cyclic homology* $HP_{\bullet}(\mathcal{A},\mathcal{B})$ is the homology of $\big(\operatorname{Tot}^{\prod}_{\bullet}\mathcal{B}C^{\mathrm{per}}_{\bullet,\bullet}(\mathcal{A},\mathcal{B}),\widetilde{b}+\widetilde{B}\big)$, where $\mathcal{B}C^{\mathrm{per}}_{p,q}(\mathcal{A},\mathcal{B})=\mathcal{B}C^{\mathrm{per}}_{p,q}(\mathcal{A})\oplus\mathcal{B}C^{\mathrm{per}}_{p,q+1}(\mathcal{B})$ and where $\widetilde{b}$, $\widetilde{B}$ are as above.

1.2. **Relative Chern character and relative pairing.** By Proposition 1.1, the relative cyclic (co)homology groups inherit a natural pairing

$$\langle -,-\rangle_{\bullet}:\ HC_{\bullet}(\mathcal{A},\mathcal{B})\times HC^{\bullet}(\mathcal{A},\mathcal{B})\to\mathbb{C}, \tag{1.10}$$

which will be called the *relative cyclic pairing.*

In Section 2.3 below we shall express the divisor flow defined by Melrose [21] in terms of such a relative pairing. In preparation for that, we recall below the definition of the (odd) Chern character in periodic cyclic homology and the affiliated transgression formulas.

Given a (Fréchet) algebra $\mathcal{A}$, and an element

$$g\in\mathrm{GL}_\infty(\mathcal{A}):=\lim_{N\to\infty}\mathrm{GL}_N(\mathcal{A}),$$

the odd Chern character is the following normalized periodic cyclic cycle:

$$\mathrm{ch}_{\bullet}(g)=\sum_{k=0}^{\infty}(-1)^k k!\,\mathrm{tr}_{2k+1}\big((g^{-1}\otimes g)^{\otimes(k+1)}\big), \tag{1.11}$$

where $(g^{-1}\otimes g)^{\otimes j}$ is an abbreviation for the j-fold tensor product

$$(g^{-1}\otimes g)\otimes\ldots\otimes(g^{-1}\otimes g),$$

and tr_k denotes the generalized trace map $\mathfrak{M}_N(\mathcal{A})^{\otimes k+1}\to\mathcal{A}^{\otimes k+1}$ (cf. [20, Def. 1.2.1]).

There is a transgression formula (cf. GETZLER [9, Prop. 3.3]) for the odd Chern character of a smooth family of invertible matrices $g_s \in \mathrm{GL}_\infty(\mathcal{A})$, $s \in [0,1]$:

$$\frac{d}{ds}\,\mathrm{ch}_\bullet(g_s) = (b+B)\ \not{c}\mathrm{h}_\bullet(g_s,\dot g_s), \tag{1.12}$$

where the secondary Chern character $\not{c}\mathrm{h}_\bullet$ is defined by

$$\begin{aligned} \not{c}\mathrm{h}_\bullet(g,h) = \mathrm{tr}_0(g^{-1}h) + \\ + \sum_{k=0}^{\infty}(-1)^{k+1}k! \sum_{j=0}^{k} \mathrm{tr}_{2k+2}\big((g^{-1}\otimes g)^{\otimes(j+1)}\otimes g^{-1}h\otimes(g^{-1}\otimes g)^{\otimes(k-j)}\big). \end{aligned} \tag{1.13}$$

Note that our sign convention differs from the one in GETZLER [9], since in Eqs. (1.8) and (1.9) we have used the sign convention of e.g. [20] for the definition of the operators b and B.

The secondary Chern character fulfills the secondary transgression formula Eq. (1.14) for smooth two-parameter families of invertibles $g_{s,t} \in \mathrm{GL}_\infty(\mathcal{A})$, $s,t \in [0,1]$. Its proof follows by straightforward computation.

$$\frac{\partial}{\partial s}\not{c}\mathrm{h}_\bullet(g,\partial_t g) - \frac{\partial}{\partial t}\not{c}\mathrm{h}_\bullet(g,\partial_s g) = (b+B)\ \not{c}\!\!\not{\mathrm{h}}_\bullet(g,\partial_s g,\partial_t g), \tag{1.14}$$

where

$$\begin{aligned} \not{c}\!\!\not{\mathrm{h}}_\bullet(g,h_1,h_2) = -\,\mathrm{tr}_1(g^{-1}h_1\otimes g^{-1}h_2) - \\ -\sum_{k=0}^{\infty}(-1)^k k! \sum_{j_1+j_2+j_3=k} \mathrm{tr}_{2k+3}\big((g^{-1}\otimes g)^{\otimes(j_1+1)}\otimes g^{-1}h_1\otimes \\ \otimes(g^{-1}\otimes g)^{\otimes j_2}\otimes g^{-1}h_2\otimes(g^{-1}\otimes g)^{\otimes j_3}\big) + \\ +\sum_{k=0}^{\infty}(-1)^k k! \sum_{j_1+j_2+j_3=k} \mathrm{tr}_{2k+3}\big((g^{-1}\otimes g)^{\otimes(j_1+1)}\otimes g^{-1}h_2\otimes \\ \otimes(g^{-1}\otimes g)^{\otimes j_2}\otimes g^{-1}h_1\otimes(g^{-1}\otimes g)^{\otimes j_3}\big). \end{aligned} \tag{1.15}$$

The transgression formula Eq. (1.12) gives rise to a relative cyclic homology class as follows. As above, let $\sigma : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism of two locally convex topological algebras. Let us call an element $a \in \mathfrak{M}_N(\mathcal{A})$ *elliptic*, if $\sigma(a)$ is invertible, or in other words lies in $\mathrm{GL}_N(\mathcal{B})$. Moreover, given a family $(a_s)_{0\le s\le 1}$ of elements of $\mathfrak{M}_N(\mathcal{A})$, we say that $(a_s)_{0\le s\le 1}$ is an *admissible elliptic path*, if each a_s is elliptic, and a_0 and a_1 are both invertible.

Proposition 1.2. *Let $(a_s)_{0\le s\le 1}$ be a smooth admissible elliptic path in $\mathfrak{M}_N(\mathcal{A})$. Then the expression*

$$\mathrm{ch}_\bullet\big((a_s)_{0\le s\le 1}\big) := \Big(\mathrm{ch}_\bullet(a_1) - \mathrm{ch}_\bullet(a_0), -\int_0^1 \not{c}\mathrm{h}_\bullet\big(\sigma(a_s),\sigma(\dot a_s)\big)\,ds\Big)$$

is well-defined and defines a relative cyclic cycle.

We then define the Chern character *of the admissible elliptic family* $(a_s)_{0\leq s\leq 1}$ *as the class of the relative cyclic cycle* $\mathrm{ch}_\bullet\big((a_s)_{0\leq s\leq 1}\big)$.

Proof. The transgression formula Eq. (1.12) entails

$$\sigma_*\big(\mathrm{ch}_\bullet(a_1)-\mathrm{ch}_\bullet(a_0)\big)=(b+B)\int_0^1 \not{c}\mathrm{h}_\bullet\big(\sigma(a_s),\sigma(\dot a_s)\big)\,ds.$$

Moreover, the odd Chern character is a cyclic cycle in $\mathcal{A}$, and therefore

$$(b+B)\big(\mathrm{ch}_\bullet(a_1)-\mathrm{ch}_\bullet(a_0)\big)=0,$$

which completes the proof. □

1.3. **Elliptic paths and relative K-theory.** With the application alluded to above in mind, it is appropriate to elaborate at this point on invertibility and ellipticity in topological algebras as well as on a representation of relative topological K-theory by homotopy classes of elliptic paths.

Assume that $\mathcal{A}$ is a topological algebra, which in particular means that the product in $\mathcal{A}$ is separately continuous. Following the presentation in Schweitzer [25], we call $\mathcal{A}$ a *good* topological algebra, if $\mathrm{GL}_1(\widetilde{\mathcal{A}})\subset\widetilde{\mathcal{A}}$ is open and the inversion is continuous, where $\widetilde{\mathcal{A}}$ denotes the smallest unital algebra containing $\mathcal{A}$. By Swan [27, Lem. 2.1], $\mathcal{A}$ being a good topological algebra implies that $\mathfrak{M}_N(\mathcal{A})$ is good as well for every $N\in\mathbb{N}^*$. Moreover, if $\sigma:\mathcal{A}\to\mathcal{B}$ is a continuous homomorphism of topological algebras with $\mathcal{A}$ a good topological algebra, then $\mathcal{B}$ is good as well by Gramsch [12, Bem. 5.4]. Thus, in this situation, the space $\mathrm{Ell}_N(\widetilde{\mathcal{A}})$ of elliptic $N\times N$ matrices with entries in $\widetilde{\mathcal{A}}$ is an open subset of $\mathfrak{M}_N(\widetilde{\mathcal{A}})$. If $\mathcal{A}$ is additionally locally convex, this implies in particular that for every continuous family $(a_s)_{0\leq s\leq 1}$ of elliptic elements in $\mathfrak{M}_N(\widetilde{\mathcal{A}})$, one can find a smooth path $(b_s)_{0\leq s\leq 1}$ which, in $\mathrm{Ell}_N(\widetilde{\mathcal{A}})$, is homotopic to $(a_s)_{0\leq s\leq 1}$ relative the endpoints. Moreover, using [12, Bem. 5.4] again, one can even show that there exists a continuous family $(q_s)_{0\leq s\leq 1}$ of *parametrices* for $(a_s)_{0\leq s\leq 1}$, which means that both a_sq_s-I and q_sa_s-I lie in the kernel of σ for each $s\in[0,1]$.

Recall that a locally convex topological algebra is called *m-convex*, if its topology is generated by submultiplicative seminorms. Next, a Fréchet subalgebra $\iota:\mathcal{A}\hookrightarrow\mathcal{A}'$ of some m-convex Fréchet algebra $\mathcal{A}'$ is called *closed under holomorphic functional calculus*, if for every $A\in\widetilde{\mathcal{A}}$ and every complex function f holomorphic on a neighborhood of the $\widetilde{\mathcal{A}'}$-spectrum of A, the element $f(A)\in\widetilde{\mathcal{A}'}$ lies in the algebra $\widetilde{\mathcal{A}}$. If $\mathcal{A}'$ is a Banach algebra (resp. C^*-algebra) and $\mathcal{A}$ is dense in $\mathcal{A}'$, then one often calls $\mathcal{A}$ a *local Banach algebra* (resp. *local C^*-algebra*) or briefly *local* in $\mathcal{A}'$, and denotes $\mathcal{A}'$ as $\bar{\mathcal{A}}$, the *completion* of $\mathcal{A}$. By a result going back to the work of Gramsch [12] and Schweitzer [25], the matrix algebra $\mathfrak{M}_N(\mathcal{A})$ over a Fréchet algebra $\mathcal{A}$ is local, if and only if $\mathcal{A}$ is. This implies in particular the following stability result.

Proposition 1.3. (Cf. [3, VI. 3]) *If $\mathcal{A}$ is a unital Fréchet algebra which is local in a Banach algebra $\bar{\mathcal{A}}$, then $\iota : \mathcal{A} \hookrightarrow \bar{\mathcal{A}}$ induces an isomorphism of topological K-theories:*

$$\iota_* : K_\bullet(\mathcal{A}) \cong K_\bullet(\bar{\mathcal{A}}).$$

Let us now briefly recall the construction of the relative K_1-group associated to a continuous unital homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ of unital Fréchet algebras. The group $K_1(\mathcal{A}, \mathcal{B})$ is defined as the kernel of the map $\mathrm{pr}_{2*} : K_1(D) \to K_1(\mathcal{A})$ induced by the projection $\mathrm{pr}_2 : D \to \mathcal{B}$ of the double

$$D = \big\{(a, b) \in \mathcal{A} \times \mathcal{A} \mid a - b \in \mathcal{J} := \operatorname{Ker} \sigma\big\}$$

onto the second coordinate. Since $K_1(\mathcal{A})$ is the quotient of $\mathrm{GL}_\infty(\mathcal{A})$ by the connected component of the identity, it is clear that $K_1(\mathcal{A}, \mathcal{B})$ can be naturally identified with the quotient of

$$G := \big\{a \in \mathrm{GL}_\infty(\mathcal{A}) \mid a = I + S \text{ with } S \in \mathcal{J}\ \big\}$$

by G_0, the connected component of the identity in G. By excision in (topological) K-theory, one knows that $K_1(\mathcal{A}, \mathcal{B})$ is naturally isomorphic to $K_1(\mathcal{J})$. We will present another identification of $K_1(\mathcal{A}, \mathcal{B})$, namely with a certain fundamental group of the space

$$\mathrm{Ell}_\infty(\mathcal{A}) := \lim_{N\to\infty} \mathrm{Ell}_N(\mathcal{A})$$

of elliptic elements of $\mathcal{A}$. Note that as an inductive limit this space carries a natural topology inherited from the Fréchet topology on $\mathcal{A}$.

Before stating the result we need to fix some more notation. For a topological space X and a subspace $Y \subset X$ we denote by $\Omega(X, Y)$ the set of continuous paths in X with endpoints in Y. If $y_0 \in Y$ is a basepoint in Y we denote by $\pi_1(X, Y; y_0)$ the relative homotopy set of homotopy classes of paths with initial point y_0 and endpoint in Y. By $\pi_1(X, Y)$ we denote the homotopy classes of paths $\sigma : [0, 1] \to X$ with $\sigma(0), \sigma(1) \in Y$. Note that for a homotopy of paths the endpoints may vary in Y. $\pi_1(X, Y)$ together with concatenation of paths is naturally a groupoid. For definiteness we denote the concatenation of paths by $*$.

On $\pi_1(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A}))$, the set of homotopy classes of admissible elliptic paths, we even have a monoid structure, $\cdot$, induced by pointwise multiplication. In fact, the two products are not independent: consider paths $f, g \in \Omega(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A}))$ and put

$$\varphi(s) = \begin{cases} 2s, & 0 \le s \le 1/2, \\ 1, & 1/2 \le s \le 1 \end{cases}, \quad \psi(s) = \begin{cases} 0, & 0 \le s \le 1/2, \\ 2s - 1, & 1/2 \le s \le 1. \end{cases} \tag{1.16}$$

Then $H(s, t) = f((1 - t)s + t\varphi(s))g((1 - t)s + t\psi(s))$ is a homotopy which shows that

$$f \cdot g \simeq (f \cdot g(0)) * (f(1) \cdot g). \tag{1.17}$$

Denote by $f_-(s) := f(1 - s)$ the inverse of the path f with respect to the concatenation product. Then $f * f_-$ is homotopic to the constant path $f(0)$.

In view of (1.17) we have $f\cdot(f(1)^{-1}f_-)\simeq f*f_-\simeq f(0)$ and thus we have shown that for every path f there is a path g such that $f\cdot g$ is homotopic to a constant path.

Finally, we call a path $f:[0,1]\longrightarrow \mathrm{GL}_\infty(\mathcal{A})$ *degenerate*. A path is degenerate, if and only if it is homotopic to a constant path. The homotopy classes of degenerate paths form a submonoid of $\pi_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$. If we mod out this submonoid of $\pi_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$ we obtain a group which we denote by $\widetilde{\pi}_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$. Observe that by Eq. (1.17) the products $*$ and $\cdot$ coincide on $\widetilde{\pi}_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$. Finally we note that thanks to the fact that we are working with paths in a stable algebra the pointwise product in $\pi_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$ may as well be represented by direct sums since $f\cdot g$ is homotopic to

$$\begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}. \tag{1.18}$$

Lemma 1.4. *The natural map*

$$\pi_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A});I)\longrightarrow\widetilde{\pi}_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A})) \tag{1.19}$$

is an isomorphism.

Remark 1.5. It is worth pointing out that this fact does not have an exact analogue in the even case (see Section 1.6 below). The reason is that $\mathrm{GL}_\infty(\mathcal{A})$ has a distinguished base point while the space of idempotents in $\mathfrak{M}_\infty(\mathcal{A})$ does not.

Proof. If $f\in\Omega(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$, then the class of the path f in $\widetilde{\pi}_1(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$ is represented by the path $f(0)^{-1}f$ starting at I. This proves surjectivity. To prove injectivity consider a path $f\in\Omega(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A}))$ starting at I such that f is homotopic to a degenerate path. Then there is a map $H:[0,1]^2\to\mathrm{Ell}_\infty(\mathcal{A})$ such that $H(s,0)=f(s)$, $H(s,1)$ is a fixed element of $\mathrm{GL}_\infty(\mathcal{A})$, and $H(0,t),H(1,t)\in\mathrm{GL}_\infty(\mathcal{A})$. Hence $\widetilde{H}(s,t):=H(0,t)^{-1}H(s,t)$ is a homotopy of paths starting at I which implements a homotopy between f and the constant I. □

Theorem 1.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Fréchet algebras and*

$$0\longrightarrow\mathcal{J}\longrightarrow\mathcal{A}\stackrel{\sigma}{\longrightarrow}\mathcal{B}\longrightarrow 0$$

an exact sequence of Fréchet algebras and unital homomorphisms such that $\mathcal{A}$ (and hence $\mathcal{B}$) is a good Fréchet algebra. Moreover, assume that $\mathcal{A}$ and $\mathcal{B}$ are both local Banach algebras such that σ extends to a continuous homomorphism $\bar{\mathcal{A}}\to\bar{\mathcal{B}}$. Then the following holds true:

(1) *For each $a_0\in\mathrm{GL}_\infty(\mathcal{A})$, the inclusion $\mathcal{A}\hookrightarrow\bar{\mathcal{A}}$ induces a natural bijection*

$$\pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}),\mathrm{GL}_\infty(\mathcal{A});a_0\big)\cong\pi_1\big(\mathrm{Ell}_\infty(\bar{\mathcal{A}}),\mathrm{GL}_\infty(\bar{\mathcal{A}});a_0\big).$$

(2) *The canonical homomorphism*

$$\kappa:\ K_1(\mathcal{A},\mathcal{B}) \to \pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A}); I\big),$$

induced by the assignment

$$I+S \mapsto \big[[0,1]\ni s\mapsto I+sS\in \mathrm{Ell}_\infty(\mathcal{A})\big]$$

is an isomorphism, which therefore induces a canonical isomorphism $K_1(\mathcal{A},\mathcal{B}) \longrightarrow \widetilde{\pi}_1(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{B}))$.

Proof. The proof of the first claim will only be sketched, since it also follows from the second and stability of K-theory for local Banach algebras. The locality implies in particular that $\mathrm{GL}_N(\mathcal{A}) = \mathrm{GL}_N(\bar{\mathcal{A}}) \cap \mathfrak{M}_N(\mathcal{A})$ for each N (see e.g. [25]). Likewise, one shows $\mathrm{Ell}_N(\mathcal{A}) = \mathrm{Ell}_N(\bar{\mathcal{A}}) \cap \mathfrak{M}_N(\mathcal{A})$. Since $\mathrm{Ell}_N(\bar{\mathcal{A}})$ is open in $\mathfrak{M}_N(\bar{\mathcal{A}})$, and the topology of $\bar{\mathcal{A}}$ is generated by a complete norm, every path in $\mathrm{Ell}_N(\bar{\mathcal{A}})$ starting at a_0 is homotopic in $\mathrm{Ell}_N(\bar{\mathcal{A}})$ to a path with values in $\mathrm{Ell}_N(\mathcal{A})$. This proves (1).

The second claim is essentially a consequence of the fact that under the assumptions made (notably locality of $\mathcal{A}$ and $\mathcal{B}$), there exists for each $b \in \mathrm{GL}_N(\mathcal{B})_I$, the connected component of the identity in $\mathrm{GL}_N(\mathcal{B})$, an invertible lift, i.e. an element $a \in \mathrm{GL}_N(\mathcal{A})_I$ with $\sigma(a) = b$ (see for example Blackadar [2, Cor. 3.4.4] or Gramsch [12, Bem. 5.4]). This lifting property holds for continuous paths as well, since with $\mathcal{A}$ being local, $\mathcal{C}^\infty([0,1],\mathcal{A}) \cong \mathcal{C}^\infty([0,1])\hat{\otimes}\mathcal{A}$ is local as well, and likewise for $\mathcal{B}$. Note also that by nuclearity of $\mathcal{C}^\infty([0,1])$ the sequence

$$0 \longrightarrow \mathcal{C}^\infty([0,1],\mathcal{J}) \longrightarrow \mathcal{C}^\infty([0,1],\mathcal{A}) \longrightarrow \mathcal{C}^\infty([0,1],\mathcal{B}) \longrightarrow 0$$

is exact as well, and that the closure of $\mathcal{C}^\infty([0,1],\mathcal{A})$ is $\mathcal{C}([0,1],\bar{\mathcal{A}})$.

After these preliminary considerations let us now prove surjectivity of κ. Let $(a_s)_{0\le s\le 1}$ be a smooth path in $\mathrm{Ell}_N(\mathcal{A})$ which starts at the identity and satisfies $a_1 \in \mathrm{GL}_N(\mathcal{A})$. By the preceding remarks there exists an invertible lift of $(\sigma(a_s))_{0\le s\le 1}$, that means a smooth path $(b_s)_{0\le s\le 1}$ of elements of $\mathrm{GL}_N(\mathcal{A})$ such that $\sigma(a_s) = \sigma(b_s)$ for all s. Hence the path $(a_s b_s^{-1})_{0\le s\le 1}$ consists of elements of the form $I + S_s$, where $S_s \in \mathcal{J}$. Moreover, since the homotopy class of $(b_s)_{0\le s\le 1}$ is trivial, $(a_s)_{0\le s\le 1}$ and $(a_s b_s^{-1})_{0\le s\le 1}$ are homotopic. Consider now the homotopy

$$h(s,t) = t\, a_s b_s^{-1} + (1-t)\big(I + s(a_1 b_1^{-1} - I)\big),$$

where $s \in [0,1]$ and $t \in [0,1]$ is the homotopy parameter. Then one has

(i) $h(0,t) = I$,
(ii) $h(1,t) = a_1 b_1^{-1}$, which is invertible,
(iii) $h(s,1) = a_s b_s^{-1}$, and
(iv) $h(s,0) = I + sS_1$ with $S_1 := (a_1 b_1^{-1} - I) \in \mathcal{J}$.

Hence $(a_s)_{0\le s\le 1}$ is homotopic to the path $h(.,0)$, which is equal to $\kappa(I+S_1)$. This shows surjectivity.

Let us finally show injectivity of κ. Assume that $\kappa(I + S)$ has trivial homotopy class. This means that there exists a homotopy h in $\mathrm{Ell}_N(\mathcal{A})$

relative $\mathrm{GL}_N(\mathcal{A})$ with $h(s,0) = I+sS$, $h(s,1) = I$ and $h(0,t) = I$. Since $\sigma\circ h$ is a homotopy of invertible elements in $\mathrm{GL}_N(\mathcal{B})$, there exists an invertible lift to $\mathcal{A}$, i.e. a homotopy g in $\mathrm{GL}_N(\mathcal{A})$ such that $\sigma \circ g = \sigma \circ h$ and

(i) $g(0,t) = g(s,1) = I$ for all $s,t \in [0,1]$.

Furthermore, one has

(ii) $g(s,0) = I + T_s$ with $T_s \in J$.

Consider now the homotopy $H(s,t) := h(s,t)\, g^{-1}(s,t)\,(I + T_{(1-t)s})$. Since $h(s,t)\, g^{-1}(s,t) - I \in \mathcal{J}$ for all s,t, one concludes that

(iii) $H(0,t) = I$,
(iv) $H(s,0) = h(s,0) = I + sS$,
(v) $H(s,1) = I$, and
(vi) $H(s,t) - I \in \mathcal{J}$ for all s,t.

Therefore, $H(s,.)$ gives rise to a homotopy in $K_1(\mathcal{A},\mathcal{B})$ between $I+S$ and I. Hence κ is injective, and the theorem follows. □

Using the preceding theorem we can now show that the Chern character from the previous section is defined even on $\pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big)$, the fundamental groupoid of the space of elliptic elements of the algebra $\mathcal{A}$ relative the invertible ones.

Theorem 1.7. *Under the assumptions of the preceding theorem let $(a_{s,t})_{0\le s,t\le 1}$ be a smooth family in $\mathfrak{M}_N(\mathcal{A})$ such that for each fixed t the family $(a_{s,t})_{0\le s\le 1}$ is a smooth admissible path. Then $\mathrm{ch}_\bullet\big((a_{s,1})_{0\le s\le 1}\big)$ and $\mathrm{ch}_\bullet\big((a_{s,0})_{0\le s\le 1}\big)$ are homologous relative cyclic cycles, and therefore the Chern character $\mathrm{ch}_\bullet$ is homotopy invariant and descends to a map on $\pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big)$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are both local Banach algebras, then*

$$\mathrm{ch}_\bullet \circ \kappa : K_1(\mathcal{A},\mathcal{B}) \to HP_1(\mathcal{A},\mathcal{B})$$

coincides, via the canonical identification $K_1(\mathcal{J}) \cong K_1(\mathcal{A},\mathcal{B})$, where $\mathcal{J} = \operatorname{Ker}\sigma$, with the standard Chern character in cyclic homology.

Proof. The secondary transgression formula Eq. (1.14) entails that

$$\begin{aligned}
&\mathrm{ch}_\bullet\big((a_{s,1})_{0\le s\le 1}\big) - \mathrm{ch}_\bullet\big((a_{s,0})_{0\le s\le 1}\big)\\
&\quad = \big(\widetilde{b} + \widetilde{B}\big)\Big(\not{c}\mathrm{h}\,\big(a_{s,1}, \partial_t a_{s,1}\big) - \not{c}\mathrm{h}\,\big(a_{s,0}, \partial_t a_{s,0}\big),\\
&\qquad\qquad - \int_0^1 \not{c}\!\!\not{c}\mathrm{h}\,\big(\sigma(a_{s,t}), \sigma(\partial_s a_{s,t}), \sigma(\partial_t a_{s,t})\big)\, ds\Big),
\end{aligned}$$

which proves the first part of the theorem. To show the second part let $[I+S] \in K_1(\mathcal{J})$ and check that

$$\mathrm{ch}_\bullet\,\big(\kappa([I+S])\big) = \big(\,\mathrm{ch}_\bullet(I) - \mathrm{ch}_\bullet(I+S), 0\big).$$

This gives the claim. □

The higher homotopy groups $\pi_n\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big)$, $n \geq 1$, satisfy a natural Bott periodicity property. In order to state it, let us assume to be given a homotopy class $[\varrho] \in \pi_2\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big)$, represented by a continuous map

$$\varrho : [0,1]^2 \to \operatorname{Ell}_N(\mathcal{A}),$$

with the additional property that

$$\varrho\big([0,1] \times \{0\}\big) = I \quad \text{and} \quad \varrho\big(\partial[0,1]^2\big) \subset \operatorname{GL}_N(\mathcal{A}).$$

Obviously, ϱ is homotopic to a continuous map $\widetilde{\varrho} : [0,1]^2 \to \operatorname{Ell}_N(\mathcal{A})$ satisfying

$$\widetilde{\varrho}\big([0,1] \times \{0\} \cup \{0,1\} \times [0,1]\big) = I \quad \text{and} \quad \widetilde{\varrho}\big(\partial[0,1]^2\big) \subset \operatorname{GL}_N(\mathcal{A}).$$

Since $\widetilde{S\mathcal{A}}$, the suspension of $\mathcal{A}$ with unit adjoined, naturally coincides with $\big\{f \in \mathcal{C}^0(S^1, \mathcal{A}) \,\big|\, f(1) \in \mathbb{C}1_{\mathcal{A}}\big\}$, the path $\widetilde{\varrho}(\cdot, t)$ is an element of $\widetilde{S\mathcal{A}}$ for each $t \in [0,1]$. Hence the map $[0,1] \ni t \mapsto \widetilde{\varrho}(\cdot, t)$ defines a homotopy class in $\pi_1\big(\operatorname{Ell}_\infty(\widetilde{S\mathcal{A}}), \operatorname{GL}_\infty(\widetilde{S\mathcal{A}}); I\big)$ which we denote by $\lambda(\varrho)$. Clearly, λ factors through $\pi_2\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big)$ and one checks immediately that the resulting map

$$\lambda : \pi_2\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big) \to \pi_1\big(\operatorname{Ell}_\infty(\widetilde{S\mathcal{A}}), \operatorname{GL}_\infty(\widetilde{S\mathcal{A}}); I\big) \tag{1.20}$$

is an isomorphism. Using suspension in K-theory one then finds

$$\begin{aligned} \pi_2\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big) &\cong \pi_1\big(\operatorname{Ell}_\infty(\widetilde{S\mathcal{A}}), \operatorname{GL}_\infty(\widetilde{S\mathcal{A}}); I\big) \cong \\ &\cong K_1(S\mathcal{A}, S\mathcal{B}) \cong K_0(\mathcal{A}, \mathcal{B}). \end{aligned} \tag{1.21}$$

By iteration of this argument and using Bott periodicity in K-theory, one obtains the corresponding periodicity property of the higher homotopy groups in the relative setting.

Theorem 1.8. *Let $\sigma : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism of good Fréchet algebras such that the assumptions of Thm. 1.6 are satisfied. Then*

$$\pi_{2n-i}\big(\operatorname{Ell}_\infty(\mathcal{A}), \operatorname{GL}_\infty(\mathcal{A}); I\big) \;=\; K_i(\mathcal{A}, \mathcal{B}), \text{ for } i = 0, 1, \text{ and all } n \geq 1.$$

1.4. **Relative cycles and their characters.** Connes' concept of a cycle over an algebra has a natural extension to the relative case. In what follows $\mathcal{A}, \mathcal{B}$ are unital (locally convex topological) $\mathbb{C}$-algebras and $\sigma : \mathcal{A} \to \mathcal{B}$ is a (continuous) unital and surjective homomorphism. We recall from [4, Part II. §3] (cf. also [11, Sec. 2]) the definition of a 'chain', which we prefer to call "relative cycle" here.

Definition 1.9. A *relative cycle* of degree k over $(\mathcal{A}, \mathcal{B})$ consists of the following data:

(1) differential graded unital algebras (Ω, d) and $(\partial\Omega, d)$ over $\mathcal{A}$ resp. $\mathcal{B}$ together with a surjective unital homomorphism $r : \Omega \to \partial\Omega$ of degree 0,
(2) unital homomorphisms $\varrho_{\mathcal{A}} : \mathcal{A} \to \Omega^0$ and $\varrho_{\mathcal{B}} : \mathcal{B} \to \partial\Omega^0$ such that $r \circ \varrho_{\mathcal{A}} = \varrho_{\mathcal{B}} \circ \sigma$,

(3) a graded trace $\int$ on Ω of degree k such that

$$\int d\omega = 0\,, \quad \text{whenever} \quad r(\omega) = 0. \tag{1.22}$$

The graded trace $\int$ induces a unique closed graded trace $\int'$ on $\partial\Omega$ of degree $k-1$, such that Stokes' formula

$$\int d\omega = \int' r\omega \quad \text{ for all } \omega \in \Omega \tag{1.23}$$

is satisfied.

A relative cycle will often be denoted in this article as a tuple $C = \big(\Omega, \partial\Omega, r, \rho_{\mathcal{A}}, \rho_{\mathcal{B}}, \int, \int'\big)$ or, more compactly, as $C = \big(\Omega, \partial\Omega, r, \int, \int'\big)$. The boundary $(\partial\Omega, d, \int')$ is just a cycle over the algebra $\mathcal{B}$. For $(\Omega, d, \int)$, this is in general not the case, unless the trace $\int$ is closed.

Note that, for the unit 1_Ω of Ω, the Leibniz rule implies $d1_\Omega = 0$. By definition we consider only cycles with $\varrho_{\mathcal{A}}$ unital. This simplifies our considerations because of $d\varrho(1_{\mathcal{A}}) = d1_\Omega = 0$. (When dealing with non-unital $\varrho_{\mathcal{A}}$ one has to accept the disturbing fact that $d\varrho_{\mathcal{A}}(1)$ might be nonzero.) If no confusion can arise, we will omit from now on the subscript of ϱ.

We next define the *character* of a relative cycle C by means of the following proposition.

Proposition 1.10. (Cf. [11, Sec. 2]) *Let C be a relative cycle of degree k over $(\mathcal{A}, \mathcal{B})$. Define $(\varphi_k, \psi_{k-1}) \in C^k(\mathcal{A}) \oplus C^{k-1}(\mathcal{B})$ as follows:*

$$\varphi_k(a_0, \dots, a_k) := \frac{1}{k!} \int \varrho(a_0) d\varrho(a_1) \dots d\varrho(a_k), \tag{1.24}$$

$$\psi_{k-1}(b_0, \dots, b_{k-1}) := \frac{1}{(k-1)!} \int' \varrho(b_0) d\varrho(b_1) \dots d\varrho(b_{k-1}). \tag{1.25}$$

Then $\operatorname{char} C := (\varphi_k, \psi_{k-1})$ *is a relative cyclic cocycle in* $\operatorname{Tot}^k_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}, \mathcal{B})$, *called the character of the relative cycle C.*

Proof. Since $(\partial\Omega, d, \int')$ is a cycle over $\mathcal{B}$, we know by the work of Connes [4] that ψ_{k-1} is a cyclic cocycle over $\mathcal{B}$, which in particular means that $(b+B)\psi_{k-1} = 0$. Hence it remains to show that

$$(b+B)\varphi_k = \sigma^* \psi_{k-1}. \tag{1.26}$$

Eq. (1.26) in fact means

$$b\varphi_k = 0, \quad B\varphi_k = \sigma^*\psi_{k-1}.$$

For simplicity we omit ϱ in the notation. Now, b-closedness follows from the trace property. Namely, for $a_0, \dots, a_{k+1} \in \mathcal{A}$ applying Leibniz rule to

$d(a_j a_{j+1})$ we find

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j \int a_0 da_1 \cdot \ldots \cdot d(a_j a_{j+1}) \cdot \ldots \cdot da_{k+1} = \\ = -\int a_0 a_1 da_2 \cdot \ldots \cdot da_{k+1} + (-1)^k \int a_0 da_1 \cdot \ldots \cdot da_k a_{k+1}\end{aligned} \tag{1.27}$$

and hence by the trace property $b\varphi_k = 0$. Using $d1 = 0$ and Stokes formula for C we find

$$\begin{aligned}
&B\varphi_k(a_0, \ldots, a_{k-1}) \\
&= \sum_{j=0}^{k-1}(-1)^{(k-1)j}\, \varphi_k(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) \\
&\qquad - \sum_{j=0}^{k-1}(-1)^{(k-1)j}\, \varphi_k(a_j, 1, a_{j+1}, \ldots, a_k, a_0, \ldots, a_{j-1}) \\
&= \frac{1}{k!}\sum_{j=0}^{k-1}(-1)^{(k-1)j} \int da_j \cdot \ldots \cdot da_{k-1}\, da_0 \cdot \ldots \cdot da_{j-1} \\
&= \frac{1}{(k-1)!}\int d\big(a_0 da_1 \cdot \ldots \cdot da_{k-1}\big) \\
&= \frac{1}{(k-1)!}\int' r(a_0) dr(a_1) \cdot \ldots \cdot dr(a_{k-1}) \\
&= \psi_{k-1}(\sigma(a_0), \ldots, \sigma(a_{k-1})),
\end{aligned} \tag{1.28}$$

where we have used that $\int$ is a graded trace. □

1.5. **Divisor flow associated to an odd relative cycle.** In the situation of Section 1.4, consider a relative cycle C of degree $2k+1$ over $(\mathcal{A}, \mathcal{B})$. Denote by $(\varphi_{2k+1}, \psi_{2k})$ the character of C as defined above, and recall that as a consequence of Proposition 1.1 we have a convenient representation of the natural pairing (1.10) in relative cyclic (co)homology.

Definition 1.11. The (*odd*) *divisor flow* of the smooth admissible elliptic path $(a_s)_{0\le s\le 1}$ with respect to the odd relative cycle C is defined to be the relative pairing between $\mathrm{ch}_\bullet\big((a_s)_{0\le s\le 1}\big)$ and the character $\mathrm{char}\, C = (\varphi_{2k+1}, \psi_{2k})$, i.e.

$$\begin{aligned}
&\mathrm{DF}_C\big((a_s)_{0\le s\le 1}\big) := \mathrm{DF}\big((a_s)_{0\le s\le 1}\big) := \\
&:= \frac{1}{(-2\pi i)^{k+1}}\langle(\varphi_{2k+1}, \psi_{2k}), \mathrm{ch}_\bullet((a_s)_{0\le s\le 1})\rangle \\
&= \frac{1}{(-2\pi i)^{k+1}}\big(\langle\varphi_{2k+1}, \mathrm{ch}_\bullet(a_1)\rangle - \langle\varphi_{2k+1}, \mathrm{ch}_\bullet(a_0)\rangle\big) \\
&\quad - \frac{1}{(-2\pi i)^{k+1}}\langle\psi_{2k}, \int_0^1 \not{c}\mathrm{h}_\bullet(\sigma(a_s), \sigma(\dot a_s))ds\rangle.
\end{aligned} \tag{1.29}$$

Simple calculations show that the partial pairings involved in the above formula can be expressed as follows:

$$\langle \varphi_{2k+1}, \mathrm{ch}_\bullet(a_s)\rangle = \frac{k!}{(2k+1)!}\int \big(a_s^{-1} da_s\big)^{2k+1}, \tag{1.30}$$

$$\begin{aligned}&\langle \psi_{2k}, \not{\mathrm{c}}\mathrm{h}_\bullet(\sigma(a_s),\sigma(\dot a_s))\rangle \\ &\quad = \frac{k!}{(2k)!}\int' (\sigma(a_s)^{-1}\sigma(\dot a_s))\big((\sigma(a_s))^{-1} d(\sigma(a_s))\big)^{2k}.\end{aligned} \tag{1.31}$$

By Theorem. 1.7, the divisor flow is a homotopy invariant, which immediately entails the following result.

Theorem 1.12. *Assume that $\sigma : \mathcal{A} \to \mathcal{B}$ is a continuous surjective unital homomorphism of Fréchet algebras, with $\mathcal{A}$ good, and let C be an odd relative cycle over $(\mathcal{A},\mathcal{B})$ with character $(\varphi_{2k+1}, \psi_{2k}) := \operatorname{char} C$. Then the divisor flow with respect to C is well-defined on $\pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big)$. Moreover, the divisor flow map*

$$\mathrm{DF}_C : \pi_1\big(\mathrm{Ell}_\infty(\mathcal{A}), \mathrm{GL}_\infty(\mathcal{A})\big) \to \mathbb{C}$$

is additive with respect to composition of paths. Finally, if in addition $\sigma : \mathcal{A} \to \mathcal{B}$ is a morphism of local Banach algebras, then the divisor flow defines a homomorphism

$$\mathrm{DF}_C : K_1(\mathcal{A},\mathcal{B}) \to \mathbb{C}.$$

1.6. **Divisor flow associated to an even relative cycle.** In this section we treat the even case. In many respects it is parallel to the odd case. Since the relative K_0-theory of algebras is quite well-covered in the literature, cf. in particular [2, Theorem 5.4.2] and Higson–Roe [14, Sec. 4.3], we will be rather brief and state the results without proof.

Throughout the entire section $\mathcal{A}$ and $\mathcal{B}$ will denote two unital Fréchet algebras, and

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0 \tag{1.32}$$

is an exact sequence of Fréchet algebras and unital homomorphisms, with $\mathcal{A}$ (and hence $\mathcal{B}$) a good Fréchet algebra.

We denote by $\mathrm{P}_\infty(\mathcal{B})$ the set of idempotents in the stable matrix algebra $\mathfrak{M}_\infty(\mathcal{B})$ and by

$$\mathrm{AP}_\infty(\mathcal{A}) := \big\{x \in \mathfrak{M}_\infty(\mathcal{A}) \,\big|\, \sigma(x) \in \mathrm{P}_\infty(\mathcal{B})\big\} \tag{1.33}$$

the set of *almost idempotents* in $\mathcal{A}$.

The relative K_0-group $K_0(\mathcal{A},\mathcal{B})$ can be described as follows: let $V(\mathcal{A},\mathcal{B})$ be the set of triples (e,f,γ), where $e,f \in \mathrm{P}_\infty(\mathcal{A})$ and $\gamma : [0,1] \to \mathrm{P}_\infty(\mathcal{B})$ is a continuous path with $\gamma(0) = \sigma(e), \gamma(1) = \sigma(f)$. A triple (e,f,γ) is *degenerate*, if γ has a continuous lift $\widetilde{\gamma} : [0,1] \longrightarrow \mathrm{P}_\infty(\mathcal{A})$ with $\sigma\circ\widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = e, \widetilde{\gamma}(1) = f$. A homotopy of triples is a continuous path (e_t, f_t, γ_t) of triples. Triples can be added in the obvious way. $K_0(\mathcal{A},\mathcal{B})$ is the group obtained from $(V(\mathcal{A},\mathcal{B}),\oplus)$ by identifying homotopic triples and dividing by the submonoid of homotopy classes of degenerate triples, cf. [14, Def.

4.3.3]. Note that [14] defines the triples differently by considering (e, f, z), where z is a Murray–von Neumann equivalence between $\sigma(e)$ and $\sigma(f)$. The resulting K-groups are the same since in the stable algebra two projections are Murray–von Neumann equivalent iff they are homotopic [2, Sec. 4.2–4.4].

By strong excision in topological K-theory [14, Theorem 4.3.8] the natural map

$$K_0(J) = K_0(\widetilde{J}, \widetilde{J}/J) \longrightarrow K_0(A, B), \quad [e] - [f] \mapsto [(f, e, 1)] \tag{1.34}$$

is an isomorphism. Note that elements of $K_0(J)$ are represented in the form $[e] - [f]$ with $e, f \in \mathrm{P}_\infty(\widetilde{J})$ such that $\sigma(e) = \sigma(f)$. Since $\mathcal{A}$ is unital, $\widetilde{J}$ is naturally a subalgebra of $\mathcal{A}$.

We now proceed to give an alternative description of the relative K_0-group, in the spirit of the discussion preceding Theorem 1.6. Let $\Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ be the set of continuous paths in $\mathrm{AP}_\infty(\mathcal{A})$ with endpoints in $\mathrm{P}_\infty(\mathcal{A})$. Direct sum makes $\Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ and also the homotopy set $\pi_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ into a monoid. A path $\gamma \in \Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ is called *degenerate*, if γ maps into $\mathrm{P}_\infty(\mathcal{A})$. The quotient of $\pi_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ by the submonoid of homotopy classes of degenerate paths, or in other words of homotopy classes of constant paths, is a group, which will be denoted $\widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$. There is an obvious homomorphism

$$\widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A})) \longrightarrow K_0(\mathcal{A}, \mathcal{B}), \quad \gamma \mapsto (\gamma(0), \gamma(1), \sigma \circ \gamma), \tag{1.35}$$

that is easily seen to be an isomorphism. Using excision we obtain the analogue of Theorem 1.6, (2) in the even case.

Theorem 1.13. *The canonical homomorphism*

$$\begin{aligned} \kappa :\ K_0(J) &\to \widetilde{\pi}_1\big(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A})\big), \\ [e] - [f] &\mapsto \big[[0,1] \ni s \mapsto (1-s)f + se \in \mathrm{AP}_\infty(\mathcal{A})\big] \end{aligned}$$

is an isomorphism.

Turning next to the even Chern character, we recall that the Chern character of an idempotent $e \in \mathrm{P}_\infty(\mathcal{A})$ is given by the formula

$$\mathrm{ch}_\bullet(e) := 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \operatorname{tr}_{2k} \Big((e - \frac{1}{2}) \otimes e^{\otimes(2k)} \Big) \in HP_0(A). \tag{1.36}$$

If $(e_s)_{0 \le s \le 1}$ is a smooth path of idempotents, then the transgression formula reads

$$\frac{d}{ds} \mathrm{ch}_\bullet(e_s) = (b + B)\, \not{\mathrm{c}}\mathrm{h}_\bullet(e_s, (2e_s - 1)\dot{e}_s); \tag{1.37}$$

here the secondary Chern character $\not{\mathrm{c}}\mathrm{h}_\bullet$ is given by

$$\not{\mathrm{c}}\mathrm{h}_\bullet(e, h) := \iota(h)\, \mathrm{ch}_\bullet(e), \tag{1.38}$$

where the map $\iota(h)$ is defined by

$$\begin{aligned}\iota(h)&(a_0 \otimes a_1 \otimes \ldots \otimes a_l)\\ &= \sum_{i=0}^{l}(-1)^i(a_0 \otimes \ldots \otimes a_i \otimes h \otimes a_{i+1} \otimes \ldots \otimes a_l).\end{aligned} \tag{1.39}$$

In analogy to Proposition 1.2 we now define the Chern character of a path $(f_s)_{0\leq s\leq 1} \in \Omega(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A}))$ by putting

$$\begin{aligned}&\mathrm{ch}_\bullet\big((f_s)_{0\leq s\leq 1}\big)\\ &\quad := \Big(\mathrm{ch}_\bullet(f_1) - \mathrm{ch}_\bullet(f_0), -\int_0^1 \not{\mathrm{c}}\mathrm{h}_\bullet\,\big(\sigma(f_s), \sigma((2f_s-1)\dot{f}_s)\big)\,ds\Big).\end{aligned} \tag{1.40}$$

By the transgression formula $\mathrm{ch}_\bullet\,\big((f_s)_{0\leq s\leq 1}\big)$ indeed is a relative cyclic cycle.

The even counterpart of the secondary transgression formula for $\not{\mathrm{c}}\mathrm{h}_\bullet$ has already appeared in Moscovici–Wu [22, Lemma 1.11], and can be stated as follows. Let $e_{s,t} \in \mathrm{P}_\infty(\mathcal{A})$, $s,t \in [0,1]$ be a smooth two parameter family of idempotents. Put

$$a_t := (2e-1)\partial_t e, \quad a_s := (2e-1)\partial_s e. \tag{1.41}$$

Then

$$\frac{\partial}{\partial s}\not{\mathrm{c}}\mathrm{h}_\bullet(e, a_t) - \frac{\partial}{\partial t}\not{\mathrm{c}}\mathrm{h}_\bullet(e, a_s) = (b+B)\not{\not{\mathrm{c}}}\mathrm{h}_\bullet(e, a_s, a_t), \tag{1.42}$$

where

$$\begin{aligned}&\not{\not{\mathrm{c}}}\mathrm{h}_\bullet(e, h_1, h_2)\\ &\quad := \iota(h_1)\iota(h_2)\,\mathrm{ch}_\bullet(e) + R_\bullet([h_1,h_2], e) - 2Q_\bullet([h_1,h_2], e).\end{aligned} \tag{1.43}$$

We omit reproducing the explicit formulas for R and Q, which are quite involved, and refer instead to [22, Prop. 1.14].

In view of the above secondary transgression formula, $\mathrm{ch}_\bullet$ is seen to be homotopy invariant. Since degenerate paths are null-homotopic and $\mathrm{ch}_\bullet$ obviously vanishes on constant paths, $\mathrm{ch}_\bullet$ does descend to a map defined on $\widetilde{\pi}_1(\mathrm{AP}_\infty(\mathcal{A}), \mathrm{P}_\infty(\mathcal{A})) \cong K_0(\mathcal{A}, \mathcal{B})$. Thus $\mathrm{ch}_\bullet \circ \kappa$ coincides with the standard Chern character on $K_0(J) \cong K_0(\mathcal{A}, \mathcal{B})$, giving the even case of Theorem 1.7.

Finally, we are going to define the analogue of the divisor flow in the even case. As in Section 1.4, we fix a relative cycle C over $(\mathcal{A}, \mathcal{B})$, this time of degree $2k$, with character $\operatorname{char} C = (\varphi_{2k}, \psi_{2k-1})$.

Let $(f_s)_{0\leq s\leq 1}$ be a smooth path in $\mathrm{AP}_\infty(\mathcal{A})$ with endpoints in $\mathrm{P}_\infty(\mathcal{A})$. Then the relative pairing between $\mathrm{ch}_\bullet\,\big((f_s)_{0\leq s\leq 1}\big)$ and the character

$(\varphi_{2k}, \psi_{2k-1})$ of C defines the *even divisor flow*

$$\begin{aligned}\mathrm{DF}_C\big((f_s)_{0\le s\le 1}\big) &:= \mathrm{DF}\big((f_s)_{0\le s\le 1}\big) := \\ &:= \frac{(-1)^{k+1}}{(2\pi i)^k}\,\big\langle\,\mathrm{char}\,C, \mathrm{ch}_\bullet((f_s)_{0\le s\le 1})\big\rangle \\ &= \frac{(-1)^{k+1}}{(2\pi i)^k}\Big(\big\langle\varphi_{2k}, \mathrm{ch}_\bullet(f_1)\big\rangle - \big\langle\varphi_{2k}, \mathrm{ch}_\bullet(f_0)\big\rangle\Big) \\ &\quad + \frac{(-1)^k}{(2\pi i)^k}\,\big\langle\psi_{2k-1}, \int_0^1 \not{c}\mathrm{h}_\bullet(\sigma(f_s), \sigma((2f_s-1)\dot f_s))ds\big\rangle.\end{aligned} \tag{1.44}$$

Explicitly, the partial pairings entering in the above formula are given by

$$\langle\varphi_{2k}, \mathrm{ch}_\bullet(f_s)\rangle = \frac{(-1)^k}{k!}\int\big(f_s - \frac{1}{2}\big)(df_s)^{2k}, \tag{1.45}$$

$$\begin{aligned}&\langle\psi_{2k-1}, \not{c}\mathrm{h}_\bullet(\sigma(f_s), \sigma((2f_s-1)\dot f_s))\rangle = \\ &\quad = \frac{(-1)^k}{(k-1)!}\int{}' \sigma((2f_s-1)\dot f_s)\big(d(\sigma(f_s))\big)^{2k-1}.\end{aligned} \tag{1.46}$$

Under the same assumptions on $\mathcal{A}$ and $\mathcal{B}$ as in Theorem 1.12, the even divisor flow then defines a homomorphism

$$\mathrm{DF}_C : K_0(\mathcal{A}, \mathcal{B}) \to \mathbb{C}. \tag{1.47}$$

2. Divisor flow on suspended pseudodifferential operators

2.1. **Pseudodifferential operators with parameter and regularized traces.** We start by establishing some specific notation and recalling a few basic notions. Let $U \subset \mathbb{R}^n$ be an open subset. We denote by $\mathrm{S}^m(U;\mathbb{R}^N)$, $m \in \mathbb{R}$, the space of symbols of Hörmander type $(1,0)$ (Hörmander [15], Grigis–Sjøstrand [13]). More precisely, $\mathrm{S}^m(U;\mathbb{R}^N)$ consists of those $a \in \mathcal{C}^\infty(U \times \mathbb{R}^N)$ such that for multi-indices $\alpha \in \mathbb{Z}_+^n, \gamma \in \mathbb{Z}_+^N$ and each compact subset $K \subset U$ we have an estimate

$$\big|\partial_x^\alpha \partial_\xi^\gamma a(x,\xi)\big| \le C_{\alpha,\gamma,K}(1+|\xi|)^{m-|\gamma|}, \quad x \in K, \xi \in \mathbb{R}^N. \tag{2.1}$$

The best constants in (2.1) provide a set of semi-norms which endow $\mathrm{S}^\infty(U;\mathbb{R}^N) := \bigcup_{m\in\mathbb{R}} S^m(U;\mathbb{R}^N)$ with the structure of a Fréchet-algebra.

A symbol $a \in S^m(U;\mathbb{R}^N)$ is called log-*polyhomogeneous* (cf. Lesch [18]), if it has an asymptotic expansion in $\mathrm{S}^\infty(U;\mathbb{R}^N)$ of the form

$$a \sim \sum_{j=0}^\infty a_j \quad \text{with } a_j = \sum_{l=0}^{k_j} b_{lj}, \tag{2.2}$$

where $a_j \in \mathcal{C}^\infty(U \times \mathbb{R}^N)$ and $b_{lj}(x,\xi) = \tilde b_{lj}(x,\xi/|\xi|)|\xi|^{m_j}\log^l|\xi|$ for $\xi \ge 1$. Here, $(m_j)_{j\in\mathbb{N}}$ is a decreasing sequence with $m \ge m_j \searrow -\infty$.

A symbol $a \in \mathrm{S}^m(U;\mathbb{R}^N)$, $m \in \mathbb{R}$, is called *classical*, if it is log-polyhomogeneous and $m_j = m - j$, $k_j = 0$ for all j. In other words, log-terms do not occur in the expansion (2.2) and the degrees of homogeneity in the asymptotic expansion decrease in integer steps.

The space of log-polyhomogeneous symbols with $m = m_0$ in the expansion (2.2) is denoted by $\mathrm{PS}^m(U;\mathbb{R}^N)$ and the space of classical symbols in $\mathrm{S}^m(U;\mathbb{R}^N)$ by $\mathrm{CS}^m(U;\mathbb{R}^N)$. Note the little subtlety that in view of the log-terms we only have $\mathrm{PS}^m \subset \bigcap_{\varepsilon>0} \mathrm{S}^{m+\varepsilon}$ but $\mathrm{PS}^m \not\subset \mathrm{S}^m$.

Fix $a \in \mathrm{S}^m(U;\mathbb{R}^n\times\mathbb{R}^p)$ (resp. $\in \mathrm{CS}^m(U;\mathbb{R}^n\times\mathbb{R}^p)$). For each fixed μ_0 we have $a(\cdot,\cdot,\mu_0) \in \mathrm{S}^m(U;\mathbb{R}^n)$ (resp. $\in \mathrm{CS}^m(U;\mathbb{R}^n)$) and hence we obtain a family of pseudodifferential operators parameterized over $\mathbb{R}^p$ by putting

$$\begin{aligned}\big[\,\mathrm{Op}(a(\mu_0))\,u\big]\,(x) &:= \big[A(\mu_0)\,u\big](x)\\ &:= \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\,a(x,\xi,\mu_0)\,\hat{u}(\xi)\,đ\xi \\ &= \int_{\mathbb{R}^n}\int_U e^{i\langle x-y,\xi\rangle}\,a(x,\xi,\mu_0)\,u(y)đyđ\xi.\end{aligned} \tag{2.3}$$

From now on, $đ$ denotes $(2\pi)^{-n/2}$-times the Lebesgue measure on $\mathbb{R}^n$.

We denote by $\mathrm{L}^m(U;\mathbb{R}^p)$ the set of all $\mathrm{Op}(a)$ (plus parameter dependent smoothing operators), where $a \in \mathrm{S}^m(U;\mathbb{R}^p)$, and by $\mathrm{CL}^m(U;\mathbb{R}^p)$ the corresponding algebra of classical parameter dependent pseudodifferential operators.

Remark 2.1. In case $p = 0$ we obtain the usual (classical) pseudodifferential operators of order m on U. Parameter dependent pseudodifferential operators play a crucial role, e.g., in the construction of the resolvent expansion of an elliptic operator (Gilkey [10]). The definition of the parameter dependent calculus is not uniform in the literature. It will be crucial in the sequel that differentiating by the parameter reduces the order of the operator. This is the convention e.g. of [10] but differs from the one in Shubin [26]. In Lesch–Pflaum [19, Sec. 3] it is shown that parameter dependent pseudodifferential operators can be viewed as translation invariant pseudodifferential operators on $U\times\mathbb{R}^p$ and therefore our convention of the parameter dependent calculus captures Melrose's suspended algebra from [21].

For a smooth manifold M and a vector bundle E over M we define the space $\mathrm{CL}^m(M,E;\mathbb{R}^p)$ of classical parameter dependent pseudodifferential operators between sections of E in the usual way by patching together local data.

The *suspension* of a (Fréchet) algebra $\mathcal{A}$ is, by definition, the algebra $C_0(\mathbb{R},\mathcal{A})$ of all continuous functions on $\mathbb{R}$ with values in $\mathcal{A}$ which vanish at infinity. However, in the pseudodifferential world the 'right' setting for considering pseudodifferential operator valued functions is the parameter dependent calculus. In this sense, $\mathrm{CL}^0(M,E;\mathbb{R})$ could be viewed as the *pseudodifferential suspension* (with unit) of the algebra $\mathrm{CL}^0(M,E)$. Indeed, it was shown by S. Moroianu [23] that the K-theory of $\mathrm{CL}^0(M,E;\mathbb{R})$ does coincide with the K-theory of $C_0(\mathbb{R}, \mathrm{CL}^0(M,E))^{++}$ (continuous functions $f : \mathbb{R} \to \mathrm{CL}^0(M,E)$ admitting a limit at $\pm\infty$ proportional to the identity).

As a side remark, we should mention that there remain some unsettled questions regarding this picture, which deserve further attention. The true suspension is not contained in $\mathrm{CL}^0(M,E;\mathbb{R})$, nor vice versa. The above mentioned K-theoretical isomorphism is abstract and not obviously canonical, although the latter is very conceivable. It would be helpful to know that an element $A \in \mathrm{CL}^0(M,E;\mathbb{R})$ has limits (in the operator norm on L^2-space) as $\mu \to \pm\infty$. However, in this respect another disconcerting thing happens. Since the (ordinary pseudodifferential) leading symbol of $A(\mu)$ is independent of μ, it is clear that $\lim_{\mu\to\infty} A(\mu) \equiv A(0)$ modulo lower order terms. On the other hand, at the K-theoretical level, $\mathrm{CL}^0(M,E;\mathbb{R})$ is the algebra of pseudodifferential operator valued functions whose limits are proportional to the identity.

We now fix a compact smooth manifold M without boundary of dimension n. Denote the coordinates in $\mathbb{R}^p$ by $\mu_1,\dots,\mu_p$ and let $\mathbb{C}[\mu_1,\dots,\mu_p]$ be the algebra of polynomials in $\mu_1,\dots,\mu_p$. By slight abuse of notation we denote by μ_j also the operator of multiplication by the j-the coordinate function. Then we have

$$\begin{aligned} \partial_j &: \mathrm{CL}^m(M,E;\mathbb{R}^p) \to \mathrm{CL}^{m-1}(M,E;\mathbb{R}^p), \\ \mu_j &: \mathrm{CL}^m(M,E;\mathbb{R}^p) \to \mathrm{CL}^{m+1}(M,E;\mathbb{R}^p). \end{aligned} \tag{2.4}$$

Also ∂_j and μ_j act naturally on $\mathrm{PS}^\infty(\mathbb{R}^p)$ and $\mathbb{C}[\mu_1,\dots,\mu_p]$ and hence on the quotient $\mathrm{PS}^\infty(\mathbb{R}^p)/\mathbb{C}[\mu_1,\dots,\mu_p]$. After these preparations we can summarize some of the results of [19].

Let E be a smooth vector bundle on M and consider $A \in \mathrm{CL}^m(M,E;\mathbb{R}^p)$ with $m+n<0$. Then for $\mu \in \mathbb{R}^p$ the operator $A(\mu)$ is trace class hence we may define the function $\mathrm{TR}(A) : \mu \mapsto \mathrm{tr}(A(\mu))$. The map TR is obviously tracial, i.e. $\mathrm{TR}(AB) = \mathrm{TR}(BA)$, and commutes with ∂_j and μ_j. In fact, the following theorem holds.

Theorem 2.2. [19, Theorems 2.2, 4.6 and Lemma 5.1] *There is a unique linear extension*

$$\mathrm{TR} : \mathrm{CL}^\infty(M,E;\mathbb{R}^p) \to \mathrm{PS}^\infty(\mathbb{R}^p)/\mathbb{C}[\mu_1,\dots,\mu_p]$$

of TR *to operators of all orders such that*

(1) $\mathrm{TR}(AB) = \mathrm{TR}(BA)$, *i.e.* TR *is tracial.*
(2) $\mathrm{TR}(\partial_j A) = \partial_j \mathrm{TR}(A)$ *for* $j = 1,\dots,p$.

This unique extension TR *satisfies furthermore:*

(3) $\mathrm{TR}(\mu_j A) = \mu_j \mathrm{TR}(A)$ *for* $j = 1,\dots,p$.
(4) $\mathrm{TR}(\mathrm{CL}^m(M,E;\mathbb{R}^p)) \subset \mathrm{PS}^{m+p}(\mathbb{R}^p)/\mathbb{C}[\mu_1,\dots,\mu_p]$.

Sketch of Proof. We briefly present two arguments which help explain why this theorem is true.

1. Taylor expansion. Given $A \in \mathrm{CL}^m(M, E; \mathbb{R}^p)$. Since differentiation by the parameter reduces the order of the operator, the Taylor expansion around 0 yields for $\mu \in \mathbb{R}^p$ (cf. [19, Prop. 4.9])

$$A(\mu) - \sum_{|\alpha| \leq N-1} \frac{(\partial_\mu^\alpha A)(0)}{\alpha!} \mu^\alpha \in \mathrm{CL}^{m-N}(M, E). \tag{2.5}$$

Hence, if N is so large that $m - N + n < 0$, then the difference (2.5) is trace class and we put

$$\mathrm{TR}(A)(\mu) := \mathrm{tr}\Big(A(\mu) - \sum_{|\alpha| \leq N-1} \frac{(\partial_\mu^\alpha A)(0)}{\alpha!} \mu^\alpha\Big) \quad \mathrm{mod}\ \mathbb{C}[\mu_1, \ldots, \mu_p]. \tag{2.6}$$

Since we mod out by polynomials, the result is in fact independent of N. This defines TR for operators of all order and the properties (1)–(3) are straightforward to verify. However, (2.5) does not give any asymptotic information and hence does not justify the fact that TR takes values in PS.

2. Differentiation and integration. Given $A \in \mathrm{CL}^m(M, E; \mathbb{R}^p)$, then $\partial^\alpha A \in \mathrm{CL}^{m-|\alpha|}(M, E, \mathbb{R}^p)$ which again is of trace class if $m - |\alpha| + \dim M < 0$. Now integrate the function $\mathrm{TR}(\partial^\alpha A)(\mu)$ back. Since we mod out polynomials this procedure is independent of α and the choice of antiderivatives. This integration procedure also explains the possible occurrence of log-terms in the asymptotic expansion and hence why TR ultimately takes values in PS. For details, see [19, Sec. 4]. □

Composing any linear functional on $\mathrm{PS}^\infty(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$ with TR yields a trace on $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$. The *regularized trace* is obtained by composition with the following regularization of the multiple integral. If $f \in \mathrm{PS}^m(\mathbb{R}^p)$ then

$$\int_{|\mu| \leq R} f(\mu) d\mu \sim_{R \to \infty} \sum_{\alpha \to -\infty} p_\alpha(\log R) R^\alpha, \tag{2.7}$$

where p_α is a polynomial of degree $k(\alpha)$. The *regularized integral* $⨍_{\mathbb{R}^p} f(\mu) d\mu$ is, by definition, the constant term in this asymptotic expansion. This notion of regularized integral is close to the Hadamard *partie finie.* It has a couple of peculiar properties, cf. [21], which were further investigated in [18, Sec. 5] and [19]. The most important features are a modified change of variables rule for linear coordinate changes and, as a consequence, the fact that Stokes' theorem does not hold in general. The failure of Stokes' theorem gives rise, as we shall see below, to a 'symbolic' *formal trace.*

Both the regularized and the formal trace are best explained in terms of differential forms on $\mathbb{R}^p$ with coefficients in $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$. Let $\Lambda^\bullet := \Lambda^\bullet(\mathbb{R}^p)^* = \mathbb{C}[d\mu_1, \ldots, d\mu_p]$ be the exterior algebra of the vector space $(\mathbb{R}^p)^*$. We put

$$\Omega_p := \mathrm{CL}^\infty(M, E; \mathbb{R}^p) \otimes \Lambda^\bullet. \tag{2.8}$$

Then, Ω_p just consists of pseudodifferential operator-valued differential forms, the coefficients of $d\mu_I$ being elements of $\mathrm{CL}^\infty(M, E; \mathbb{R}^p)$.

For a p-form $A(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p$ we define the *regularized trace* by

$$\overline{\mathrm{TR}}(A(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p) := \int\!\!\!\!\!\!-_{\mathbb{R}^p} \mathrm{TR}(A)(\mu) d\mu_1 \wedge \ldots \wedge d\mu_p. \tag{2.9}$$

On forms of degree less than p the regularized trace is defined to be 0. $\overline{\mathrm{TR}}$ is a graded trace on the differential algebra $(\Omega_p,\, d)$. In general, $\overline{\mathrm{TR}}$ is not closed. However, its boundary,

$$\widetilde{\mathrm{TR}} := d\overline{\mathrm{TR}} := \overline{\mathrm{TR}} \circ d\,,$$

called the *formal trace*, is a closed graded trace of degree $p-1$. It is shown in [19, Prop. 5.8], [21, Prop. 6] that $\widetilde{\mathrm{TR}}$ is *symbolic*, i.e. it descends to a well-defined closed graded trace of degree $p-1$ on

$$\partial\Omega_p := \mathrm{CL}^\infty(M, E; \mathbb{R}^p)/\mathrm{CL}^{-\infty}(M, E; \mathbb{R}^p) \otimes \Lambda^\bullet. \tag{2.10}$$

Denoting by r the quotient map $\Omega_p \to \partial\Omega_p$ we see that Stokes' formula with 'boundary'

$$\overline{\mathrm{TR}}(d\omega) = \widetilde{\mathrm{TR}}(r\omega) \tag{2.11}$$

now holds by construction for any $\omega \in \Omega$. In sum, we have constructed a cycle with boundary

$$C^p_{\mathrm{reg}} := (\Omega_p,\, \partial\Omega_p,\, r,\, \varrho,\, \overline{\mathrm{TR}},\, \widetilde{\mathrm{TR}}) \tag{2.12}$$

in the sense of Definition 1.9.

Remark 2.3. To shorten the notation, in what follows we suppress the subscript $_p$ in the above cycle, and use the abbreviation $\mathrm{C}\mathcal{S}^\infty(M, E; \mathbb{R}^p) := \mathrm{CL}^\infty(M, E; \mathbb{R}^p)/\mathrm{CL}^{-\infty}(M, E; \mathbb{R}^p)$. Moreover, if no confusion can arise, we will often write CL^m_p for $\mathrm{CL}^m(M, E; \mathbb{R}^p)$, CS^m_p for $\mathrm{CS}^m(M, E; \mathbb{R}^p)$, and finally $\mathrm{C}\mathcal{S}^m_p$ for $\mathrm{C}\mathcal{S}^m(M, E; \mathbb{R}^p)$.

2.2. **The Fréchet structure of the suspended algebra.** For each fixed $m \in \mathbb{Z}$, the space $\mathrm{CL}^m(M, E; \mathbb{R}^p)$, $p \in \mathbb{N}$, carries a natural Fréchet topology, which will be described below. To keep the notation simple, we assume that the vector bundle E is trivial. For the construction of the Fréchet structure on CL^m_p it will be convenient to use the global symbol calculus for pseudodifferential operators developed by Widom [28] (see also [24]). This requires to fix a Riemannian metric on M. Let d be the geodesic distance with respect to this Riemannian metric, and ε the corresponding injectivity radius. Then choose a smooth function $\chi : \mathbb{R}M \to [0,1]$ such that $\chi(s) = 0$ for $s \geq \varepsilon$ and $\chi(s) = 1$ for $s \leq \frac{3}{4}\varepsilon$. Put $\psi(x, y) = \chi(d^2(x, y))$ for $x, y \in M$ and $\psi(v) = \chi(\pi(v), \exp v)$ for $v \in TM$, where π is the projection of the (co)tangent bundle. Then ψ is a cut-off function around the diagonal of $M \times M$ resp. around the zero-section of TM. With these data let us define two maps, namely the *symbol map* $\mathrm{Symb} : \mathrm{CL}^m_p \to \mathrm{CS}^m_p$ and the *operator map* $\mathrm{Op} : \mathrm{CS}^m_p \to \mathrm{CL}^m_p$, by putting for $A \in \mathrm{CL}^m_p$, $a \in \mathrm{CS}^m_p$ and $\mu \in \mathbb{R}^p$:

$$\mathrm{Symb}(A) : T^*M \times \mathbb{R}^p \to \mathbb{C},\ (\xi,\mu) \mapsto \big[A(\mu)\psi(.,\pi(\xi))\, e^{i\langle \exp^{-1}_{\pi(\xi)}(.),\xi\rangle}\big](\pi(\xi)),$$

$$\mathrm{Op}(a)(\mu)u : M \to \mathbb{C},\ x \mapsto \int_{T^*_xM}\int_{T_xM} a(\xi,\mu)\, e^{-i\langle v,\xi\rangle}\, \psi(v)\, u(\exp v)\, đv\, đ\xi.$$

The maps Op and Symb are quasi-inverse to each other, which means that

$$A - \mathrm{Op}(\mathrm{Symb}(A)) \in \mathrm{CL}^{-\infty}_p \ \text{ and } \ a - \mathrm{Symb}(\mathrm{Op}(a)) \in \mathrm{CS}^{-\infty}_p \tag{2.13}$$

for all $A \in \mathrm{CL}^m_p$ and $a \in \mathrm{CS}^m_p$. As a consequence Symb induces a natural topological isomorphism

$$\mathrm{C}\mathcal{S}^\infty_p = \mathrm{CL}^\infty_p / \mathrm{CL}^{-\infty}_p \longrightarrow \mathrm{CS}^\infty_p \,/\, \mathrm{CS}^{-\infty}_p, \tag{2.14}$$

whose inverse is induced by Op.

Now choose a finite open covering $\mathcal{U} = (U_j)_{j\in J}$ of M together with a subordinate partition of unity $(\varphi_j)_{j\in J}$ such that for every $j \in J$ there exists a coordinate system $x_{(j)} : U_j \to \mathbb{R}^n$. For $N \in \mathbb{N}$ and $A \in \mathrm{CL}^m_p$ then put:

$$\begin{aligned} q_{m,N}(A) := \sum_{j\in J} \sum_{|\alpha|,|\beta|,|\gamma|\le N} \sup_{\substack{(x,\xi)\in T^*M\\ \mu\in\mathbb{R}^p}} & (1+|\xi|^2+|\mu|^2)^{\frac{m-|\beta|-|\gamma|}{2}} \cdot \\ & \cdot \varphi_j(x)\, \partial^\alpha_{x_{(j)}} \partial^\beta_{\xi_{(j)}} \partial^\gamma_\mu \,\mathrm{Symb}(A)(x,\xi,\mu), \\ \widetilde{q}_{m,N}(A) := \sum_{j\in J} \sum_{\substack{|\alpha|,|\beta|,|\gamma|\le N\\ 0\le l\le N}} \sup_{\substack{(x,\xi)\in T^*M\\ \mu\in\mathbb{R}^p}} & (1+|\mu|^2)^{\frac{l}{2}} \cdot \\ & \cdot \varphi_j(x)\, \partial^\alpha_{x_{(j)}} \partial^\beta_{\xi_{(j)}} \partial^\gamma_\mu K_{A(\mu)-\mathrm{Op}(\mathrm{Symb}(A))(\mu)}(x,\xi), \end{aligned}$$

where K_B denotes the Schwartz kernel of a pseudodifferenetial operator B. One checks immediately that the $q_{m,N}$ and $\widetilde{q}_{m,N}$ are seminorms which turn CL^m_p into a Fréchet space and even a Fréchet algebra, in case $m = 0$. With respect to this Fréchet topology on CL^m_p, the subspace $\mathrm{CL}^{-\infty}_p$ is a closed subspace, hence the quotient space $\mathrm{C}\mathcal{S}^m_p$ inherits a Fréchet topology from CL^m_p, and we get an exact sequence of Fréchet spaces

$$0 \longrightarrow \mathrm{CL}^{-\infty}_p \longrightarrow \mathrm{CL}^m_p \xrightarrow{\ \sigma\ } \mathrm{C}\mathcal{S}^m_p \longrightarrow 0\,. \tag{2.15}$$

Moreover, $\mathrm{CL}^{-\infty}_p$ is a local Banach algebra whose closure is the p-fold suspension $\mathcal{C}_0(\mathbb{R}^p,\mathcal{K})$ of the algebra of compact operators $\mathcal{K}$. If $m = 0$, the algebra CL^m_p is a local Banach algebra as well (cf. [23, Sec. 5]).

Let us note that, in a similar way, one obtains for each $m \in \mathbb{Z}$ the short (and topologically split) exact sequence of Fréchet spaces

$$0 \longrightarrow \mathrm{CL}^{m-1}_p \longrightarrow \mathrm{CL}^m_p \xrightarrow{\ \sigma_m\ } \mathcal{C}^\infty(S^*_pM) \longrightarrow 0\,, \tag{2.16}$$

where $S^*_pM := \{(\xi,\mu) \in T^*M \times \mathbb{R}^p \mid |x|^2 + |\mu|^2 = 1\}$ denotes the *p-suspended cosphere bundle*, and σ_m is the principal symbol map for parametric pseudodifferential operators of order m.

2.3. **Divisor flows as a relative cyclic pairing.** Let us consider now the suspended algebra $\mathrm{CL}^\infty_{2k+1}$ of pseudodifferential operators on a compact manifold M with values in a bundle $E \to M$. Since it gives rise to the short exact sequence of local Banach algebras (2.15), we are in a relative situation and can apply the abstract results of the first part to the cycle with boundary $(\Omega, \partial\Omega, r, \varrho, \overline{\mathrm{TR}}, \widetilde{\mathrm{TR}})$ defined in Section 2.1.

Thus, let us assume to be given a smooth family $A_s \in \mathrm{CL}^\infty_{2k+1}$, $s \in [0,1]$, of elliptic operators of some fixed order $m \in \mathbb{N}$, such that A_0 and A_1 are invertible. According to Prop. 1.2, the family A_s gives rise to the relative cyclic cycle

$$\Big(\mathrm{ch}_\bullet(A_1) - \mathrm{ch}_\bullet(A_0),\, -\int_0^1 \not{c}\mathrm{h}\left(\sigma(A_s)\,\sigma(\dot{A}_s)\right) ds\Big).$$

As explained in Section 1.5, one can pair this cyclic cycle with the character $(\varphi_{2k+1}, \psi_{2k}) := \operatorname{char} C^{2k+1}_{\mathrm{reg}}$ of the relative cycle $(\Omega_{2k+1},\, \partial\Omega_{2k+1},\, r,\, \varrho,\, \overline{\mathrm{TR}},\, \widetilde{\mathrm{TR}})$ to obtain the divisor flow of the suspended algebra of pseudodifferential operators. By Eqs. (1.30) and (1.31)

$$\begin{aligned}
&\langle \varphi_{2k+1}, \mathrm{ch}_\bullet(A)\rangle = \frac{k!}{(2k+1)!}\overline{\mathrm{TR}}\left((A^{-1}dA)^{2k+1}\right),\\
&\langle \psi_{2k},\, \not{c}\mathrm{h}_\bullet(\sigma(A), \sigma(\dot{A}))\rangle \\
&\qquad = \frac{k!}{(2k)!}\widetilde{\mathrm{TR}}\left(\sigma(A)^{-1}\sigma(\dot{A})\big(\sigma(A)^{-1}d\sigma(A)\big)^{2k}\right),
\end{aligned} \tag{2.17}$$

and therefore the divisor flow has the form

$$\begin{aligned}
&\mathrm{DF}\left((A_s)_{0\le s\le 1}\right) = \\
&= \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\Big(\overline{\mathrm{TR}}\left((A_1^{-1}dA_1)^{2k+1}\right) - \overline{\mathrm{TR}}\left((A_0^{-1}dA_0)^{2k+1}\right)\Big)\\
&\quad - \frac{k!}{(-2\pi i)^{k+1}(2k)!}\int_0^1 \widetilde{\mathrm{TR}}\left(\sigma(A_s)^{-1}\,\sigma(\partial_s A_s)\,\left(\sigma(A_s)^{-1}\,d\sigma(A_s)\right)^{2k}\right) ds.
\end{aligned} \tag{2.18}$$

For $k = 0$ this is precisely the divisor flow originally defined by Melrose [21], while for $k > 0$ it gives its generalization by Lesch-Pflaum [19, Prop. 6.3].

It will be convenient for what follows to introduce one additional piece of notation: for every $m \in \mathbb{N}$, $\mathrm{Ell}^m_\infty(\mathrm{CL}^\infty_p)$ will denote the space of elliptic elements of order m in CL^∞_p, and $\pi_1\big(\mathrm{Ell}^m_\infty(\mathrm{CL}^\infty_p), \mathrm{GL}_\infty(\mathrm{CL}^\infty_p)\big)$ stands for the fundamental groupoid of the space of elliptic elements of order m relative the invertible ones.

Theorem 1.12 specializes to the present situation and reads as follows.

Theorem 2.4. *For each $m \in \mathbb{N}$, the odd divisor flow defines a map*

$$\mathrm{DF} : \pi_1\big(\mathrm{Ell}^m_\infty(\mathrm{CL}^\infty_{2k+1}), \mathrm{GL}_\infty(\mathrm{CL}^\infty_{2k+1})\big) \to \mathbb{C},$$

which is additive with respect to composition of paths. Furthermore, it induces a homomorphism from $K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1})$ *to* $\mathbb{C}$.

As will be seen shortly, the latter homomorphism actually establishes the isomorphism $K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1}) \cong \mathbb{Z}$. (Cf. Prop. 2.11 below).

Remark 2.5. It is fairly obvious that for a fixed $A \in \mathrm{GL}_\infty(\mathrm{CL}^\infty_p)$ of order m the space $\pi_1\big(\mathrm{Ell}^m_\infty(\mathrm{CL}^\infty_{2k+1}), \mathrm{GL}_\infty(\mathrm{CL}^\infty_{k+1}); A\big)$ of homotopy classes starting at A is naturally isomorphic to the set $\pi_1\big(\mathrm{Ell}_\infty(\mathrm{CL}^0_{2k+1}), \mathrm{GL}_\infty(\mathrm{CL}^0_{k+1}); I\big)$ via the map

$$(A_s)_{0\leq s\leq 1} \mapsto (A^{-1}A_s)_{0\leq s\leq 1}.$$

2.4. **Log-additivity and integrality of the divisor flow.** Let $A, B \in \mathrm{CL}^p_{2k+1}$ be invertible. The expression

$$\overline{\mathrm{TR}}\big((A^{-1}dA)^{2k+1}\big) \tag{2.19}$$

occurring in the definition of the divisor flow has been investigated in [21] in the case $k = 0$ and in [19] in general. In the case $k = 0$, (2.19) extends to a homomorphism from $K_1^{\mathrm{alg}}(\mathrm{CL}^\infty_1) \to \mathbb{Z}$. In the case $k = 1$ it was shown in [19, Remark 6.8] that

$$\begin{aligned} &\mathrm{TR}(((AB)^{-1}d(AB))^3) - \mathrm{TR}((A^{-1}dA)^3) - \mathrm{TR}((B^{-1}dB)^3) \\ &\quad = -3d\,\mathrm{TR}(A^{-1}dA \wedge dBB^{-1}), \end{aligned} \tag{2.20}$$

and so the difference on the left hand side is symbolic. For $k \geq 2$, it can also be shown that the difference on the left hand side of (2.20) equals $d\,\mathrm{TR}$ of a (noncommutative) polynomial in A, B, dA, dB and hence is symbolic, too. This being said, the following result may come as a surprise.

Theorem 2.6. *Let* $A_s \in \mathrm{CL}^m_{2k+1}$ *and* $B_s \in \mathrm{CL}^n_{2k+1}$ *with* $s \in [0,1]$ *and* $m, n \in \mathbb{Z}$ *be admissible paths of elliptic elements. Then we have*

$$\mathrm{DF}\,\big((A_sB_s)_{0\leq s\leq 1}\big) = \mathrm{DF}\,\big((A_s)_{0\leq s\leq 1}\big) + \mathrm{DF}\,\big((B_s)_{0\leq s\leq 1}\big). \tag{2.21}$$

For $k = 0$ the theorem is trivial, while for $k = 1$ it follows from (2.20). The general case is more subtle though, and its proof will be accomplished in a series of steps.

1. Reduction to constant path.

Lemma 2.7. (1) *If* $B \in \mathrm{GL}_\infty\left(\mathrm{CL}^n_{2k+1}\right)$*, then the* constant *path* B *has vanishing divisor flow.*

(2) *Let* $(A_s) \in \mathrm{CL}^m_{2k+1}$ *be an admissible path of elliptic elements and let* $B \in \mathrm{CL}^0_{2k+1}$ *be invertible. Then*

$$\mathrm{DF}\,\big((A_sB)_{0\leq s\leq 1}\big) = \mathrm{DF}\,\big((BA_s)_{0\leq s\leq 1}\big) = \mathrm{DF}\,\big((A_s)_{0\leq s\leq 1}\big).$$

Proof. The first claim is obvious from the definition. The second claim follows from the well-known homotopy

$$\begin{pmatrix} A_s & 0 \\ 0 & A_0 \end{pmatrix}\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{2.22}$$

which shows that $\begin{pmatrix} A_sB & 0 \\ 0 & A_0 \end{pmatrix}$ and $\begin{pmatrix} A_s & 0 \\ 0 & A_0B \end{pmatrix}$ are homotopic. From the homotopy invariance of the divisor flow and (1) we infer that $\mathrm{DF}\big((A_sB)_{0\le s\le 1}\big) = \mathrm{DF}\big((A)_{0\le s\le 1}\big)$. The proof of $\mathrm{DF}\big((BA_s)_{0\le s\le 1}\big) = \mathrm{DF}\big((A_s)_{0\le s\le 1}\big)$ is analogous. □

We emphasize that this argument fails if the degree of B is different from 0, because then (2.22) is not a valid homotopy of admissible paths!

Lemma 2.8. *To prove Theorem* 2.6 *it suffices to show that for each* $n \in \mathbb{R}$ *there exists an invertible* $B_n \in \mathrm{CL}^n_{2k+1}$ *such that for admissible paths* $A_s \in \mathrm{CL}^0_{2k+1}$, $s \in [0,1]$ *with* $A_0 = I$ *one has*

$$\mathrm{DF}\big((A_sB_n)_{0\le s\le 1}\big) = \mathrm{DF}\big((B_nA_s)_{0\le s\le 1}\big) = \mathrm{DF}\big((A_s))_{0\le s\le 1}\big). \tag{2.23}$$

Proof. In the square $[0,1]\times[0,1]$ the path $s \mapsto (s,s)$ is homotopic to the concatenation of the paths $s \mapsto (s,0)$ and $s \mapsto (1,s)$. Consequently the path $s \mapsto A_sB_s$ is homotopic to the concatenation of the paths $s \mapsto A_sB_0$ and $s \mapsto A_1B_s$. Hence we are reduced to the case that one of the two paths is constant.

Now suppose that the condition of the lemma is fulfilled, that $(A_s)_{0\le s\le 1}$ is an admissible path of elliptic elements in CL^m_{2k+1} and that $B' \in \mathrm{CL}^n_{2k+1}$ is invertible. Then applying Lemma 2.7 we find

$$\begin{aligned} \mathrm{DF}\big((A_sB')_{0\le s\le 1}\big) &= \mathrm{DF}\big((A_sA_0^{-1}B_{m+n})_{0\le s\le 1}(B_{m+n}^{-1}A_0B')\big) \\ &= \mathrm{DF}\big((A_sA_0^{-1}B_{m+n})_{0\le s\le 1}\big) = \mathrm{DF}\big((A_sA_0^{-1})_{0\le s\le 1}\big). \end{aligned} \tag{2.24}$$

The right hand side is independent of B'. Eq. (2.24) applies in particular to $B' = I$ and hence $\mathrm{DF}\big((A_sA_0^{-1})_{0\le s\le 1}\big) = \mathrm{DF}\big((A_s)_{0\le s\le 1}\big)$. The proof for $\mathrm{DF}\big((B'A_s)_{0\le s\le 1}\big)$ works exactly the same way. □

2. Reduction to the finite-dimensional case. By Lemma 2.8 and the homotopy invariance of the divisor flow we need to prove (2.23) for one representative $(A_s)_{0\le s\le 1}$ of each class in the relative K_1-group $K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1})$ and one constant invertible B_n for each n. Next we are going to choose convenient B_n. Choose a Riemannian metric on M, a hermitian metric on E as well as a metric connection on E. Denote by Δ^E the connection Laplacian acting on sections of E. Then Δ^E is a non-negative self-adjoint elliptic differential operator. Put

$$B_n(\mu) := \big(\Delta^E + I + |\mu|^2\big)^{q/2}. \tag{2.25}$$

Then B_n is an invertible element in CL^n_{2k+1}. It has the nice property that the operators $B_n(\mu)$ commute and have a joint spectral decomposition. Since there is an inclusion $I + \mathcal{C}^\infty_c(\mathbb{R}^{2k+1}, \mathfrak{M}_\infty(\mathbb{C})) \subset I + \mathrm{CL}^{-\infty}_{2k+1}$ which induces an isomorphism in K-theory, we may represent a class in $K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1})$ in the following form:

$$A_s(\mu) = I + sg(\mu)P, \tag{2.26}$$

where P is a spectral projection of Δ^E of sufficiently high rank and $g : \mathbb{R}^{2k+1} \longrightarrow \operatorname{End}(\operatorname{Im} P)$ is a smooth function with compact support such that $I + g(\mu)$ is invertible.

The spectral projection P reduces the operator A_s as well as B_n. Although P is not in the parametric calculus, a direct calculation of TR shows that the divisor flow of $(A_s)_{0\le s\le 1}$ equals the divisor flow of the *finite-rank* family $(PA_sP)_{0\le s\le 1}$. Also, the divisor flow of $(A_sB)_{0\le s\le 1}$ equals the divisor flow of the finite-rank family $(PA_sBP)_{0\le s\le 1}$.

Hence we are reduced to prove Theorem 2.6 in the case $M = \{\mathrm{pt}\}$:

3. The finite-dimensional case. Consider $M = \{\mathrm{pt}\}$. Then

$$\mathcal{A}^m := \mathrm{CL}^m(\{\mathrm{pt}\}, \mathbb{C}^N; \mathbb{R}^{2k+1}) = \mathrm{CS}^m(\{\mathrm{pt}\} \times \mathbb{R}^{2k+1}) \otimes \operatorname{End} \mathbb{C}^N \tag{2.27}$$

is precisely the algebra of $\operatorname{End} \mathbb{C}^N$-valued symbols of Hörmander type $(1, 0)$.

The divisor flow makes perfectly sense for admissible paths of elliptic elements of $\mathcal{A}^\infty$. The Theorem 2.6 will now follow from Lemma 2.8, the following lemma, and the subsequent remark.

Lemma 2.9. *Let $A_s \in \mathcal{A}^m, B_s \in \mathcal{A}^n$, $s \in [0, 1]$ be admissible paths of elliptic elements. Furthermore, assume that B_s is of the form $B_s(\mu) = f_s(\mu) \otimes I_{\mathbb{C}^N}$, i.e. $(B_s)_{0\le s\le 1}$ is a path of central elements of $\mathcal{A}$. Then Eq.* (2.21) *holds for A and B.*

Remark 2.10. As a constant path we may e.g. choose

$$B(\mu) := (1 + |\mu|^2)^{q/2} \otimes I_{\mathbb{C}^N}. \tag{2.28}$$

Proof. As noted at the beginning of this section, the case $k = 0$ is trivial, so we assume $k \ge 1$ and abbreviate $\omega_s := A_s^{-1} dA_s$ and $\eta_s := B_s^{-1} dB_s$. Then ω_s satisfies

$$d\omega_s^{2l-1} = -\omega_s^{2l}. \tag{2.29}$$

Since B_s is a scalar function, we find

$$d\eta_s = 0, \qquad (\omega_s + \eta_s)^{2l} = \omega_s^{2l}, \tag{2.30}$$

$$\eta_s^2 = 0, \qquad (\omega_s + \eta_s)^{2l+1} = \omega_s^{2l+1} - d(\omega_s^{2l-1} \wedge \eta_s), \tag{2.31}$$

and furthermore

$$(A_sB_s)^{-1} d(A_sB_s) = \omega_s + \eta_s. \tag{2.32}$$

Next we deal with the ingredients of the divisor flow.

$$\begin{aligned}
&\overline{\mathrm{TR}}(\omega_1^{2k+1}) - \overline{\mathrm{TR}}(\omega_0^{2k+1}) - \overline{\mathrm{TR}}((\omega_1 + \eta_1)^{2k+1}) + \overline{\mathrm{TR}}((\omega_0 + \eta_0)^{2k+1}) \\
&\quad = \overline{\mathrm{TR}}(\omega_1^{2k-1} \wedge \eta_1) - \widetilde{\mathrm{TR}}(\omega_0^{2k-1} \wedge \eta_0) \\
&\quad = \int_0^1 \frac{d}{ds} \widetilde{\mathrm{TR}}\Big((\sigma(A_s)^{-1} d\sigma(A_s))^{2k-1} \wedge \sigma(B_s)^{-1} d\sigma(B_s) \Big)\, ds.
\end{aligned} \tag{2.33}$$

We abbreviate $\widetilde{\omega}_s := \sigma(A_s)^{-1}d\sigma(A_s), \widetilde{\eta}_s := \sigma(B_s)^{-1}d\sigma(B_s)$. Taking into account that B is central, one sees as in the proof of [19, Prop. 6.3] that

$$\begin{aligned} \frac{d}{ds}&\widetilde{\mathrm{TR}}(\widetilde{\omega}_s^{2k-1}\wedge\widetilde{\eta}_s) \\ &= \widetilde{\mathrm{TR}}\Big(d\Big((\sigma(A_s)^{-1}\sigma(\partial_s A_s))\widetilde{\omega}_s^{2k-2}\wedge\eta_s\Big)\Big) \\ &\quad + \widetilde{\mathrm{TR}}\Big(\widetilde{\omega}_s^{2k-1}\wedge d\big(\sigma(B_s)^{-1}\sigma(\partial_s B_s)\big)\Big). \end{aligned} \tag{2.34}$$

The first summand on the right vanishes because $\widetilde{\mathrm{TR}}$ is a closed trace. For the second summand we find

$$\begin{aligned} \widetilde{\mathrm{TR}}&\Big(\widetilde{\omega}_s^{2k-1}\wedge d\big(\sigma(B_s)^{-1}\sigma(\partial_s B_s)\big)\Big) \\ &= \widetilde{\mathrm{TR}}\Big(\sigma(B_s)^{-1}\sigma(\partial_s B_s)d\widetilde{\omega}_s^{2k-1}\Big) \\ &= -\widetilde{\mathrm{TR}}\Big(\sigma(B_s)^{-1}\sigma(\partial_s B_s)\widetilde{\omega}_s^{2k}\Big) = 0, \end{aligned} \tag{2.35}$$

since the one-form ω_s commutes with $\sigma(B_s)^{-1}\sigma(\partial_s B_s)$ and the exponent $2k$ is even.

In sum, the left hand side of Eq. (2.33) vanishes. Moreover, since $\eta_s^2 = 0$ we have $\overline{\mathrm{TR}}(\eta_s^{2k+1}) = 0$ for $k \geq 1$. Thus, in the expansion of the difference

$$\mathrm{DF}\left((A_sB_s)_{0\leq s\leq 1}\right) - \mathrm{DF}\left((A_s)_{0\leq s\leq 1}\right) - \mathrm{DF}\left((B_s)_{0\leq s\leq 1}\right) \tag{2.36}$$

the terms involving $\overline{\mathrm{TR}}$ add up to 0. Furthermore, since $\eta_s^2 = 0$ we have

$$\widetilde{\mathrm{TR}}\left(\sigma(B_s)^{-1}\,\sigma(\partial_s B_s)\left(\sigma(B_s)^{-1}\,d\sigma(B_s)\right)^{2k}\right) = 0.$$

Finally, since $(\omega_s + \eta_s)^{2k} = \omega_s^{2k}$, we have

$$\begin{aligned} \widetilde{\mathrm{TR}}&\left(\sigma(A_sB_s)^{-1}\,\sigma(\partial_s(A_sB_s))\left(\sigma(A_sB_s)^{-1}\,d\,\sigma(A_sB_s)\right)^{2k}\right) \\ &= \widetilde{\mathrm{TR}}\left(\sigma(A_s)^{-1}\,\sigma(\partial_s A_s)\,\widetilde{\omega}_s^{2k}\right) \end{aligned}$$

in view of Eq. (2.35). This proves that in the expression for Eq. (2.36) the terms involving $\widetilde{\mathrm{TR}}$ add up to 0. The Lemma is proved. □

As a consequence of the additivity and of the K-theoretic interpretation of the divisor flow we can now prove its integrality. This generalizes [21, Prop. 8].

Proposition 2.11. *The divisor flow defined on the homotopy groupoid* $\pi_1\big(\mathrm{Ell}^m_\infty(\mathrm{CL}^0_{2k+1}), \mathrm{GL}_\infty(\mathrm{CL}^0_{2k+1})\big)$ *assumes integer values. Moreover, it induces an isomorphism*

$$K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1}) \xrightarrow{\simeq} \mathbb{Z}.$$

Proof. Let $(A_s)_{0\le s\le 1}$ be an admissible path of elliptic elements of CL^m_{2k+1}. By the additivity and by the fact that the divisor flow of a constant path vanishes we find

$$\mathrm{DF}\left((A_s)_{0\le s\le 1}\right) = \mathrm{DF}\left((A_s A_0^{-1})_{0\le s\le 1}\right) \tag{2.37}$$

hence it suffices to prove integrality for admissible paths of 0^{th}-order elements starting at I. By Theorem 1.6 it remains to prove integrality of the divisor flow for standard paths in $I+\mathrm{CL}^{-\infty}_{2k+1}$.

Consider an (ordinary) pseudodifferential projection $P \in \mathrm{CL}^0(M,E)$ of finite rank and a smooth map

$$g : \mathbb{R}^{2k+1} \longrightarrow \mathrm{GL}(\operatorname{Im} P) \text{ with } \lim_{|\mu|\to\infty} g(\mu) = I. \tag{2.38}$$

In other words, this means that g is an element of the algebra obtained by adjoining a unit to $\mathcal{C}^\infty_0(\mathbb{R}^{2k+1}, \mathrm{GL}(\operatorname{Im} P))$. The map

$$T(\mu) := g(\mu)P + I - P \tag{2.39}$$

is in $I+\mathrm{CL}^{-\infty}_{2k+1}$. Put

$$A_s := (1-s)I + sT, \quad 0\le s\le 1. \tag{2.40}$$

Then $(A_s)_{0\le s\le 1}$ is a smooth family of elliptic elements in CL^0_{2k+1} with $A_0 = I$ and A_1 invertible.

Every element of $K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1})$ can be represented in this way. Indeed, by excision and the well-known fact that dense subalgebras which are stable under holomorphic functional calculus have the same K-theory as the original algebra

$$\begin{aligned} K_1(\mathrm{CL}^0_{2k+1}, \mathrm{C}\mathcal{S}^0_{2k+1}) &= K_1(\mathrm{CL}^{-\infty}_{2k+1}) = K_1(\mathcal{C}^\infty_0(\mathbb{R}^{2k+1}, \mathfrak{M}_\infty(\mathbb{C}))) \\ &= K_1(\mathcal{C}^\infty_0(\mathbb{R}^{2k+1}, \mathbb{C})) \cong \mathbb{Z}. \end{aligned} \tag{2.41}$$

To achieve the proof it will suffice to show that the divisor flow of (A_s) is integral and that there exists a path (A_s) of the form (2.40) and of divisor flow 1.

Since $\sigma(A_s) = 1$ and $A_0 = I$, the divisor flow equals

$$\begin{aligned} \mathrm{DF}\left((A_s)_{0\le s\le 1}\right) &= (-2\pi i)^{-(k+1)} \langle \varphi_{2k+1}, \mathrm{ch}_\bullet(A_1)\rangle \\ &= \frac{k!}{(-2\pi i)^{k+1}(2k+1)!} \int_{\mathbb{R}^{2k+1}} \operatorname{tr}(g(\mu)^{-1} dg(\mu))^{2k+1} d\mu. \end{aligned} \tag{2.42}$$

The latter is precisely the odd Chern character of g [9, Prop. 1.4]. The odd Chern character is known to be an isomorphism from $K_1(\mathcal{C}_0(\mathbb{R}^{2k+1}))$ onto $\mathbb{Z}$, and so we reach the desired conclusion. □

2.5. **Compatibility with Bott periodicity.** Recall the exact sequence (2.15) of Fréchet spaces

$$0 \longrightarrow \mathrm{CL}_p^{-\infty} \longrightarrow \mathrm{CL}_p^m \xrightarrow{\sigma} \mathrm{C}\mathcal{S}_p^m \longrightarrow 0 .$$

As has been mentioned, $\mathrm{CL}_p^{-\infty}$ is nothing but the p-fold smooth suspension of the algebra $\mathrm{CL}^{-\infty}(M,E)$ of smoothing operators acting on sections of E. In turn, $\mathrm{CL}^{-\infty}(M,E)$ is a local Banach algebra whose closure is the algebra of compact operators, hence the K-groups of $\mathrm{CL}^{-\infty}(M,E)$ are naturally isomorphic to those of $\mathbb{C}$. For $m=0$ the above exact sequence consists of algebras, hence by excision,

$$K_i(\mathrm{CL}^0_{2k+i},\, \mathrm{C}\mathcal{S}^0_{2k+i}) \cong K_i(\mathrm{CL}^{-\infty}_{2k+i}) \xrightarrow{\simeq} \mathbb{Z}\,, \qquad i=0,1. \tag{2.43}$$

The latter isomorphism is of course the classical Bott isomorphism, but it can also be regarded as being induced by the restriction of the divisor flow pairing. More precisely, the divisor flow pairing acquires the following topological interpretation.

Theorem 2.12. *The divisor flow pairing with the character* $\operatorname{char} C^p_{\mathrm{reg}}$ *of the relative cycle* $(\Omega_p,\, \partial\Omega_p,\, r,\, \varrho,\, \overline{\mathrm{TR}},\, \widetilde{\mathrm{TR}})$, $p=2k+i>0$, *implements the Bott isomorphism at the relative K-theory level,*

$$K_i(\mathrm{CL}^0_{2k+i},\, \mathrm{C}\mathcal{S}^0_{2k+i}) \xrightarrow{\simeq} \mathbb{Z}\,, \qquad i=0,1\,,$$

in a manner compatible with the Bott suspension.

Proof. The fact that (2.43) is an isomorphism follows in the odd case from Proposition 2.11 and its proof. What remains to be proved is the compatibility with the Bott suspension, which automatically implies the even case. To this end, we will relate the construction (cf. (2.8)–(2.11)) of our relative cycle $(\Omega, \partial\Omega, r, \varrho, \overline{\mathrm{TR}}, \widetilde{\mathrm{TR}})$ to the work of Elliott–Natsume–Nest [8] on the cyclic cohomology of one-parameter crossed products.

Assume to be given a Fréchet algebra $\mathcal{A}$ and consider the trivial $\mathbb{R}$-action on $\mathcal{A}$. Then the smooth crossed product $\mathcal{A} \rtimes \mathbb{R}$ is, via Fourier transform, isomorphic to the smooth suspension $\mathcal{A} \otimes \mathscr{S}(\mathbb{R}) = \mathscr{S}(\mathbb{R}, \mathcal{A})$. Here $\mathscr{S}(\mathbb{R})$ denotes the space of Schwartz functions. For the definition of the smooth suspension see [8, 2.4]. Note that $\mathscr{S}(\mathbb{R})$ is nuclear which makes dealing with topological tensor products more convenient. The smooth suspension is a local Banach algebra with completion $C_0(\mathbb{R}) \otimes \overline{\mathcal{A}}$, hence we have natural isomorphisms

$$K_i(\mathscr{S}(\mathbb{R}, \mathcal{A})) \cong K_i(C_0(\mathbb{R}) \otimes \overline{\mathcal{A}}) \cong K_{i+1}(\overline{\mathcal{A}}) \cong K_{i+1}(\mathcal{A}). \tag{2.44}$$

In [8] the authors produce a natural map

$$\# : HC^\bullet(\mathcal{A}) \longrightarrow HC^{\bullet+1}(\mathscr{S}(\mathbb{R}, \mathcal{A})) \tag{2.45}$$

which commutes with the periodicity operator S and defines isomorphisms

$$HP^\bullet(\mathcal{A}) \longrightarrow HP^{\bullet+1}(\mathscr{S}(\mathbb{R}, \mathcal{A})). \tag{2.46}$$

Furthermore, if

$$\beta : K_i(\mathcal{A}) \longrightarrow K_{i+1}(\mathscr{S}(\mathbb{R}, \mathcal{A})) \tag{2.47}$$

denotes the Bott suspension isomorphism, then one has for $\varphi \in Z^k_\lambda(\mathcal{A})$, $[e] \in K_k(\mathcal{A})$

$$\langle \varphi, [e] \rangle = \langle \# \varphi, \beta[e] \rangle. \tag{2.48}$$

This means that via the natural pairing between periodic cyclic cohomology and K-theory # corresponds to the Bott isomorphism. □

We will now describe the map # in more detail on the level of cycles such that the relation between # and our suspended relative cycle becomes apparent. Obviously, smooth suspensions correspond to crossed products with trivial $\mathbb{R}$-actions. In our case it will be more convenient to deal with the algebra $\mathscr{S}(\mathbb{R})$ with the product of pointwise multiplication. In the more general situation of [8] one has to deal with $\mathscr{S}_*(\mathbb{R})$, that is $\mathscr{S}(\mathbb{R})$ equipped with the convolution product. This has to be taken into account when one compares the formulas in [8] with ours.

We consider a locally convex cycle $(\Omega, d_{\mathcal{A}}, \tau)$ of degree n over the Fréchet algebra $\mathcal{A}$. That is $(\Omega, d_{\mathcal{A}})$ is a locally convex differential graded algebra with a continuous closed graded trace τ of degree n together with a continuous homomorphism $\mathcal{A} \longrightarrow \Omega^0$. We single out an important example:

Example 2.13 (The fundamental cycle of $\mathscr{S}(\mathbb{R}^p)$)**.** Consider the algebra $\mathscr{S}(\mathbb{R}^p)$ and put

$$\mathscr{E}^\bullet := \mathscr{S}(\mathbb{R}^p) \otimes \Lambda^\bullet, \tag{2.49}$$

(cf. (2.8)). This means that $\mathscr{E}^\bullet$ just consists of differential forms on $\mathbb{R}^p$ with coefficients in $\mathscr{S}(\mathbb{R}^p)$. With the natural identification $\mathscr{S}(\mathbb{R}^p) \cong \mathscr{E}^0$, $d_{\mathbb{R}^p}$ the exterior derivative, and $\tau = \int_{\mathbb{R}^p}$ we obtain a cycle $(\mathscr{E}, d_{\mathbb{R}^p}, \int_{\mathbb{R}^p})$ of degree p over $\mathscr{S}(\mathbb{R}^p)$.

By [8, Cor. 5.3] the character of this cycle gives an isomorphism $HP^{p \bmod 2}(\mathscr{S}(\mathbb{R}^p)) \cong \mathbb{C}$. Furthermore $HP^{p+1 \bmod 2}(\mathscr{S}(\mathbb{R}^p)) \cong 0$. Therefore, it is appropriate to call $(\mathscr{E}, d_{\mathbb{R}^p}, \int_{\mathbb{R}^p})$ the *fundamental cycle* of $\mathscr{S}(\mathbb{R}^p)$.

Turning back to the cycle $C = (\Omega, d_{\mathcal{A}}, \tau)$ we construct a cycle of degree $n + p$, the *cup-product* of $(\Omega, d_{\mathcal{A}}, \tau)$ by the fundamental cycle of $\mathscr{S}(\mathbb{R}^p)$, as follows:

$$(\Omega^\bullet \cup \mathscr{E}^\bullet)^k := \bigoplus_{i+j=k} \Omega^i \otimes \mathscr{E}^j. \tag{2.50}$$

Note that with the natural identification $\Omega^i \otimes \mathscr{S}(\mathbb{R}^p) = \mathscr{S}(\mathbb{R}^p, \Omega^i)$, elements of $\Omega^i \otimes \mathscr{E}^j$ correspond to differential j-forms on $\mathbb{R}^p$ with coefficients in $\mathscr{S}(\mathbb{R}^p, \Omega^i)$. We will adopt this point of view if convenient.

The differential on $(\Omega^\bullet \cup \mathscr{E}^\bullet)^k$ is defined as

$$\widetilde{d}(a \otimes \omega) = d_{\mathcal{A}} a \otimes \omega + (-1)^i a \otimes d_{\mathbb{R}^p} \omega \tag{2.51}$$

for $a \in \Omega^i$ and $\omega \in \mathscr{E}^j$. Respectively, if $f \in \mathscr{S}(\mathbb{R}^p, \Omega^i)$, then

$$\widetilde{d}(f d\mu_I) = (d_{\mathcal{A}} \circ f) d\mu_I + (-1)^i d_{\mathbb{R}^p}(f d\mu_I). \tag{2.52}$$

Finally, we define a continuous linear functional $\widetilde{\tau}$ on $(\Omega^\bullet \cup \mathscr{E}^\bullet)^{n+p}$ by putting

$$\widetilde{\tau}(f d\mu_1 \wedge \ldots \wedge d\mu_p) = \int_{\mathbb{R}^p} \tau(f(\mu)) d\mu \tag{2.53}$$

and $\widetilde{\tau}_{|\Omega^i \otimes \mathscr{E}^j} = 0$ for $j < p$. One then checks immediately

Lemma 2.14. *The cup-product* $(\Omega, d_{\mathcal{A}}, \tau) \cup (\mathscr{E}^\bullet, d, \int_{\mathbb{R}^p}) := (\Omega^\bullet \cup \mathscr{E}^\bullet, \widetilde{d}, \widetilde{\tau})$ *is a cycle of degree* $n+p$ *over* $\mathscr{S}(\mathbb{R}^p, \mathcal{A})$*. The fundamental cycle of* $\mathscr{S}(\mathbb{R}^p)$ *is the* p*-fold cup product of the fundamental cycle of* $\mathscr{S}(\mathbb{R})$ *by itself.*

From [8, 3.3 and 3.7] we now infer

Proposition 2.15. *The Elliott–Natsume–Nest-map*

$$\underbrace{\# \circ \ldots \circ \#}_{p \text{ times}} : \; Z^n_\lambda(\mathcal{A}) \longrightarrow Z^{n+p}_\lambda(\mathscr{S}(\mathbb{R}^p, \mathcal{A})) \tag{2.54}$$

is given by assigning to the character of a cycle C *of degree* n *over* $\mathcal{A}$ *the character of the cycle* $C \cup (\mathscr{E}^\bullet, d, \int_{\mathbb{R}^p})$ *of degree* $n+p$ *over* $\mathscr{S}(\mathbb{R}^p, \mathcal{A})$*.*

Consider now our relative cycle $(\Omega, \partial\Omega, r, \varrho, \overline{\mathrm{TR}}, \widetilde{\mathrm{TR}})$ (cf. (2.8)–(2.11)). Restricting it to the algebra of parameter dependent smoothing operators we obtain a cycle of degree p, $(\Omega, d, \overline{\mathrm{TR}})$, over $\mathrm{CL}^{-\infty}(M, E; \mathbb{R}^p) = \mathscr{S}(\mathbb{R}^p, \mathrm{CL}^\infty(M, E))$. The L^2-trace gives rise to a canonical cycle of degree 0, $(\mathrm{CL}^{-\infty}(M, E), \mathrm{tr})$, over $\mathrm{CL}^{-\infty}(M, E)$ and the cycle $(\Omega, d, \overline{\mathrm{TR}})$ is just the cup product of $(\mathrm{CL}^{-\infty}(M, E), \mathrm{tr})$ by the fundamental cycle over $\mathscr{S}(\mathbb{R}^p)$. The interesting fact we showed is that the cycle

$$(\Omega, d, \overline{\mathrm{TR}}) = (\mathrm{CL}^{-\infty}(M, E), \mathrm{tr}) \cup (\mathscr{E}^\bullet, d_{\mathbb{R}^p}, \int_{R^p}) \tag{2.55}$$

extends through the exact sequence (2.15) to a relative cycle $(\Omega_p, \partial\Omega_p, r, \varrho, \overline{\mathrm{TR}}, \widetilde{\mathrm{TR}})$ over the full algebra of parameter dependent pseudodifferential operators. This was possible since the integrated L^2-trace $\mathrm{tr} \cup \int_{\mathbb{R}^p}$ has a tracial extension, namely the regularized trace $\overline{\mathrm{TR}}$.

3. Spectral flow as divisor flow

As Melrose pointed out from its very inception (cf. [21, p. 543]), the divisor flow has properties which closely parallel those of the spectral flow. The goal of this last section is to show that this analogy can actually be upgraded to a precise relationship, which moreover makes sense in every dimension, regardless of parity.

Since at the K-theoretical level the distinction between "even" and "odd" is encoded by the Clifford algebra, we preface the discussion by briefly recalling some basic facts about it, which will also allow us to establish the notation.

First, we recall that the Clifford algebra $\mathbb{C}\ell_p$ is the universal C^*-algebra generated by p unitaries $e_1, \ldots, e_p$ subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}.$$

For $p = 2k+1$, $\mathbb{C}\ell_p$ has a unique irreducible representation, which in standard form is realized by a $*$-homomorphism $c : \mathbb{C}\ell_{2k+1} \to \mathfrak{M}_{2^k}(\mathbb{C})$ satisfying

$$c(i^{k+1}\, e_1 \, \cdots \, e_{2k+1}) \;=\; \mathrm{Id}\,.$$

When $p = 2k$ the standard Clifford representation $c : \mathbb{C}\ell_{2k} \to \mathfrak{M}_{2^k}(\mathbb{C})$ sends the volume element into a grading operator,

$$c(i^k\, e_1 \, \cdots \, e_{2k}) \;=\; \gamma\,, \qquad \text{with} \quad \gamma^2 = \mathrm{Id} \quad \text{and} \quad \gamma^* = \gamma\,,$$

which gives a decomposition

$$\mathbb{C}^{2^k} = \Delta^+ \oplus \Delta^-, \quad \text{where } \Delta^\pm = \mathrm{Ker}(\gamma \mp \mathrm{Id})\,, \tag{3.1}$$

such that

$$c(\mu) = \begin{pmatrix} 0 & -c^+(\mu)^* \\ c^+(\mu) & 0 \end{pmatrix}, \qquad \text{for } \mu \in \mathbb{R}^p\,.$$

Here we have been making the identification

$$\mu \;\equiv\; \sum_{j=1}^{p} \mu_j\, e_j\,, \qquad \text{for } \mu \in \mathbb{R}^p\,, \tag{3.2}$$

which will remain in use for all dimensions $p \in \mathbb{N}$, regardless of parity.

3.1. **Odd case.** Starting with the odd case, we consider a smooth path of first order self-adjoint elliptic *differential* operators acting between sections of E, $(D_s)_{0\leq s\leq 1}$, with D_0, D_1 invertible, and define its *p-fold suspension* as

$$\mathcal{D}^\pm_{p,s}(\mu) := D_s \otimes I_{\mathbb{C}^{2^k}} \pm c(\mu)\,, \qquad \mu \in \mathbb{R}^p\,, \quad p = 2k+1\,.$$

Theorem 3.1. *The suspended family* $\big(\mathcal{D}^\pm_{p,s}\big)_{0\leq s\leq 1}$ *is a smooth path of elliptic elements in* CL^1_p, *with* $\mathcal{D}_{p,0}$, $\mathcal{D}_{p,1}$ *invertible, and its divisor flow* DF *is related to the spectral flow* SF *of the original family by the identity*

$$\mathrm{DF}\big((\mathcal{D}^\pm_{p,s})_{0\leq s\leq 1}\big) = \pm\, \mathrm{SF}\big((D_s)_{0\leq s\leq 1}\big)\,. \tag{3.3}$$

Proof. Since D_s is a *differential operator*, the parametric complete symbol of $D_s \otimes I_{\mathbb{C}^{2^k}} \pm c(\mu)$ is a polynomial in the cotangent variables and μ, and therefore $\mathcal{D}^\pm_{p,s} \in \mathrm{CL}^1_{2k+1}$.

If $\sigma_{D_s}(x,\xi)$ denotes the leading symbol of D_s then $\sigma_{D_s}(x,\xi) \otimes I_{\mathbb{C}^{2^k}} \pm c(\mu)$ is the parametric leading symbol of $\mathcal{D}_{p,s}$. Thus $\mathcal{D}^\pm_{p,s}$ is elliptic since D_s is elliptic. Also, we infer from

$$(\mathcal{D}^\pm_{p,s})^* \mathcal{D}^\pm_{p,s} \;=\; \Big(D_s^2 + |\mu|^2\Big) \otimes I_{\mathbb{C}^{2^k}} \tag{3.4}$$

that D_s is invertible if and only if $\mathcal{D}_{p,s}$ is invertible in CL^1_{2k+1}.

The identity Eq. (3.3) now follows by considering the *spectral η-invariant* of D_s (cf. [10, Sec. 1.13 and 3.8]). Recall that the η-function of D_s,

$$\eta(z; D_s) := \sum_{\lambda\in\operatorname{spec} D_s} (\operatorname{sgn}\lambda)|\lambda|^{-z}, \tag{3.5}$$

extends meromorphically to $\mathbb{C}$ with isolated simple poles. Furthermore, 0 is not a pole and one defines the reduced η-invariant of D_s as

$$\widetilde{\eta}(D_s) := \frac{1}{2}\big(\eta(0; D_s) + \dim\operatorname{Ker} D_s\big). \tag{3.6}$$

As a function of s, the reduced η-invariant may have integer jumps. Hence, mod $\mathbb{Z}$, the reduced η-invariant depends smoothly on s. The net number of the integer jumps equals the *spectral flow* [17, Lemma 3.4]. Namely,

$$\operatorname{SF}\big((D_s)_{0\le s\le 1}\big) = \widetilde{\eta}(D_1) - \widetilde{\eta}(D_0) - \int_0^1 \frac{d}{ds}(\widetilde{\eta}(D_s) \bmod \mathbb{Z})\, ds. \tag{3.7}$$

For the moment, consider $s \in J$ in an open subinterval of $[0, 1]$, where D_s is invertible. Then by [19, Prop. 6.6] we have for $s \in J$

$$\widetilde{\eta}(D_s) = \pm\frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\overline{\operatorname{TR}}\Big(\big((\mathcal{D}^{\pm}_{p,s})^{-1} d\mathcal{D}^{\pm}_{p,s}\big)^{2k+1}\Big). \tag{3.8}$$

The right hand side of Eq. (3.8) has been studied extensively in [19]. Up to a sign, it was called the (parametric) η-invariant of the family $\mathcal{D}^{\pm}_{p,s}$. The variation formula for the parametric η-invariant [19, Prop. 6.3], which also follows from the transgression formula Eq. (1.12) and Eqs. (1.30), (1.31), yields for $s \in J$

$$\begin{aligned}&\frac{d}{ds}\big(\widetilde{\eta}(D_s) \bmod \mathbb{Z}\big)\\ &= \pm\frac{k!}{(-2\pi i)^{k+1}(2k)!}\widetilde{\operatorname{TR}}\Big(\sigma(\mathcal{D}^{\pm}_{p,s})^{-1}\sigma(\partial_s\mathcal{D}^{\pm}_{p,s})\big(\sigma(\mathcal{D}^{\pm}_{p,s})^{-1} d\sigma(\mathcal{D}^{\pm}_{p,s})\big)^{2k}\Big),\end{aligned} \tag{3.9}$$

where σ is the symbol map from the exact sequence (2.15). We have used the fact that $\widetilde{\operatorname{TR}}$ is symbolic, cf. Eq. (2.10). Hence Eq. (3.9) makes sense if D_s is just elliptic. A priori, however, Eq. (3.9) does only hold for D_s invertible. If D_s is invertible except for finitely many $s \in [0, 1]$ then (3.9) does hold on $[0, 1]$ for continuity reasons. The general case is treated by the usual general position argument: indeed there exists a sequence $\varepsilon_j > 0$, $\varepsilon_j \to 0$ such that $D_s + \varepsilon_j$ is invertible except for finitely many values of s. Hence (3.9) holds for $D_s + \varepsilon_j$ and for continuity reasons it does hold for D_s and all $s \in [0, 1]$.

Inserting Eqs. (3.8) and (3.9) into the formula (1.29) for the divisor flow (cf. also Eq. (2.18)) gives

$$\begin{aligned}
&\mathrm{DF}\big((\mathcal{D}^{\pm}_{p,s})_{0\le s\le 1}\big)\\
&= \frac{k!}{(-2\pi i)^{k+1}(2k+1)!}\Big(\overline{\mathrm{TR}}\big(((\mathcal{D}^{\pm}_{p,1})^{-1}d\mathcal{D}^{\pm}_{p,1})^{2k+1}\big) - \overline{\mathrm{TR}}\big(((\mathcal{D}^{\pm}_{p,0})^{-1}d\mathcal{D}^{\pm}_{p,0})^{2k+1}\big)\Big)\\
&\quad - \frac{k!}{(-2\pi i)^{k+1}(2k)!}\int_0^1 \widetilde{\mathrm{TR}}\left(\big((\mathcal{D}^{\pm}_{p,s})^{-1}\,\partial_s\mathcal{D}^{\pm}_{p,s}\big)\big((\mathcal{D}^{\pm}_{p,s})^{-1}\,d\mathcal{D}^{\pm}_{p,s}\big)^{2k}\right)ds,\\
&= \pm\left(\widetilde{\eta}(D_1) - \widetilde{\eta}(D_0) - \int_0^1 \frac{d}{ds}(\widetilde{\eta}(D_s) \bmod \mathbb{Z})\,ds\right)\\
&= \pm\,\mathrm{SF}\big((D_s)_{0\le s\le 1}\big).
\end{aligned}$$

□

3.2. **Even case.** We begin by considering a single operator $D : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$, assumed to be a first order invertible self-adjoint differential operator acting on a vector bundle $E \to M$. Its spectral η-function then satisfies

$$\eta(s; D) = \mathrm{tr}\,\big(D(D^2)^{-(s+1)/2}\big). \tag{3.10}$$

In [19, Prop. 6.5], it has been shown for odd p that

$$\eta(D) := \eta(0; D) = \frac{\Gamma\big(\frac{p+1}{2}\big)}{\pi^{(p+1)/2}}\,\overline{\mathrm{TR}}\big(D(D^2 + |\,\mathrm{Id}_{\mathbb{R}^p}|^2)^{-(p+1)/2}\big), \tag{3.11}$$

where for reasons of clarity the symbol-valued trace on $\mathrm{CL}(M, E; \mathbb{R}^p)$ has been denoted with a subscript, i.e. by TR. In the following we will prove that this formula also holds true for even p. To this end we first have to recall some analytic tools, cf. [19, Sec. 6].

If $f : (0,\infty) \to \mathbb{C}$ denotes a locally integrable function with log-polyhomogeneous asymptotic expansions for $x \to 0$ and $x \to \infty$, we put

$$⨍_0^\infty f(r)dr := \operatorname*{LIM}_{\varepsilon\to 0}\int_\varepsilon^1 f(r)dr + \operatorname*{LIM}_{R\to\infty}\int_1^R f(r)dr, \tag{3.12}$$

where LIM stands for the constant term in the corresponding asymptotic expansion. Using this regularized integral the following formula has been shown in [19, Prop. 6.5] for $z \in \mathbb{N}^*$:

$$\begin{aligned}
&\binom{z-1-\frac{s+1}{2}}{z-1}\eta(s; D) =\\
&= 2\,\frac{\sin\pi\frac{s+1}{2}}{\pi}⨍_0^\infty r^{2z-2-s}\,\mathrm{TR}_1\,\big(D(D^2 + |\,\mathrm{Id}_{\mathbb{R}}|^2)^{-z}\big)(r)dr.
\end{aligned}$$

Note that both sides are meromorphic in $s \in \mathbb{C}$. Expressing the binomial coefficient and $\frac{\pi}{\sin\pi y} = \Gamma(y)\Gamma(1-y)$ in terms of Γ-functions, one obtains

$$\begin{aligned}
&\eta(s; D) = \\
&= \frac{2\Gamma(z)}{\Gamma\big(\frac{s+1}{2}\big)\Gamma\big(z - \frac{s+1}{2}\big)}⨍_0^\infty r^{2z-2-s}\,\mathrm{TR}_1\,\big(D(D^2 + |\,\mathrm{Id}_{\mathbb{R}}|^2)^{-z}\big)(r)dr.
\end{aligned} \tag{3.13}$$

By the argument for the proof of [19, Prop. 6.5], it is clear that this formula actually holds for all real $z > \frac{1}{2}$ up to a discrete set (and that the right side actually extends to a meromorphic function in z). Now observe that under the assumption $z > \frac{1}{2}$ the left side is regular at $s = 0$ and that the factor $\frac{\Gamma(z)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma(z-\frac{s+1}{2})}$ is both regular at $s = 0$ and non-vanishing. Hence the regularized integral in Eq. (3.13) is regular at $s = 0$ as well, and we have for $z > \frac{1}{2}$,

$$\eta(D) = \frac{2\Gamma(z)}{\sqrt{\pi}\,\Gamma(z-\frac{1}{2})} \int\!\!\!\!\!\!-_0^\infty r^{2z-2}\,\mathrm{TR}_1\left(D(D^2+|\,\mathrm{Id}_{\mathbb{R}}\,|^2)^{-z}\right)(r)dr. \tag{3.14}$$

Using this, and the rotation invariance of the function (defined on $\mathbb{R}^p$) $\overline{\mathrm{TR}}\big(D(D^2+|\,\mathrm{Id}_{\mathbb{R}^p}\,|^2)^{-\frac{p+1}{2}}\big)$, one obtains:

$$\begin{aligned}
&\overline{\mathrm{TR}}\left(D(D^2+|\,\mathrm{Id}_{\mathbb{R}^p}\,|^2)^{-\frac{p+1}{2}}\right)\\
&\quad= \int\!\!\!\!\!\!-_{\mathbb{R}^p} \mathrm{TR}\left(D(D^2+|\,\mathrm{Id}_{\mathbb{R}^p}\,|^2)^{-\frac{p+1}{2}}\right)(\mu)\,d\mu \qquad (3.15)\\
&\quad= \operatorname*{LIM}_{R\to\infty} \frac{p\,\pi^{p/2}}{\Gamma\big(\frac{p}{2}+1\big)} \int_0^R r^{p-1}\,\mathrm{TR}_1\left(D(D^2+|\,\mathrm{Id}_{\mathbb{R}}\,|^2)^{-\frac{p+1}{2}}\right)(r)\,dr\\
&\quad= \frac{p\,\pi^{p/2}}{\Gamma\big(\frac{p}{2}+1\big)} \int\!\!\!\!\!\!-_0^\infty r^{p-1}\,\mathrm{TR}_1\left(D(D^2+|\,\mathrm{Id}_{\mathbb{R}}\,|^2)^{-\frac{p+1}{2}}\right)(r)\,dr\\
&\quad= \frac{\pi^{(p+1)/2}}{\Gamma\big(\frac{p+1}{2}\big)}\,\eta(D).
\end{aligned}$$

This shows that Eq. (3.11) holds in all dimensions p.

We now use the standard Clifford representation c to define the p-fold suspension of the operator D in the even case $p = 2k$ as the parametric differential operator

$$\mathcal{D}_{2k}(\mu) := \gamma\left(D\otimes I_{\mathbb{C}^{2^k}} + c(\mu)\right) = \begin{pmatrix} D & -c^+(\mu)^* \\ -c^+(\mu) & -D \end{pmatrix}. \tag{3.16}$$

By construction, $\mathcal{D}_{2k}$ is an element of $\mathrm{CL}^1(M, E\otimes\mathbb{C}^{2^k};\mathbb{R}^{2k})$. From the invertibility of D, it follows that the operator $\mathcal{D}_{2k}$ is invertible. Moreover, $\mathcal{D}_{2k}(\mu)^2 = \left(D^2+|\mu|^2\right)\otimes I_{\mathbb{C}^{2^k}}$ is diagonal with respect to the decomposition (3.1). Hence

$$\mathcal{Q} := \left(\mathcal{D}_{2k}^2\right)^{1/2} = \left(D^2+|\,\mathrm{Id}_{\mathbb{R}^{2k}}\,|^2\right)^{1/2}\otimes I_{\mathbb{C}^{2^k}} \in \mathrm{CL}^1(M, E\otimes\mathbb{C}^{2^k};\mathbb{R}^{2k})$$

is invertible and

$$\mathcal{P} := \frac{1}{2}\left(I - \mathcal{Q}^{-1}\mathcal{D}_{2k}\right) \in \mathrm{CL}^0(M, E\otimes\mathbb{C}^{2^k};\mathbb{R}^{2k}) \tag{3.17}$$

is an idempotent.

Let us now determine $\overline{\mathrm{TR}}\Big(\big(\mathcal{P}-\frac{1}{2}\big)(d\mathcal{P})^{2k}\Big)$. To this end observe first that $\mathcal{D}_{2k}$ commutes with $\mathcal{Q}$ and that

$$d\mathcal{D}_{2k} = \sum_{j=1}^{2k} \gamma\, c(e_j)\, d\mu_j. \tag{3.18}$$

One also checks immediately

$$d\mathcal{Q}\wedge d\mathcal{Q} = 0, \quad \mathcal{Q}^{-1} d\mathcal{D}_{2k} = (d\mathcal{D}_{2k})\mathcal{Q}^{-1}, \quad d\mathcal{Q}\wedge d\mathcal{D}_{2k} + d\mathcal{D}_{2k}\wedge d\mathcal{Q} = 0.$$

These relations entail the following two chains of equalities:

$$\begin{aligned} d\mathcal{P}\wedge d\mathcal{P} &= \frac{1}{4} d(\mathcal{Q}^{-1}\mathcal{D}_{2k})\wedge d(\mathcal{Q}^{-1}\mathcal{D}_{2k}) = \\ &= \frac{1}{4}\big(\mathcal{Q}^{-2}(d\mathcal{Q})\mathcal{D}_{2k}\wedge \mathcal{D}_{2k}\mathcal{Q}^{-2}(d\mathcal{Q}) - \mathcal{Q}^{-2}(d\mathcal{Q})\mathcal{D}_{2k}\wedge\mathcal{Q}^{-1}d\mathcal{D}_{2k} - \\ &\quad - \mathcal{Q}^{-1}d\mathcal{D}_{2k}\wedge\mathcal{D}_{2k}\mathcal{Q}^{-2}(d\mathcal{Q}) + \mathcal{Q}^{-1}d\mathcal{D}_{2k}\wedge\mathcal{Q}^{-1}d\mathcal{D}_{2k}\big) \\ &= \frac{1}{4}\mathcal{D}_{2k}^{-2} d\mathcal{D}_{2k}\wedge d\mathcal{D}_{2k}, \end{aligned}$$

and

$$\begin{aligned} (d\mathcal{P})^{2k} &= 4^{-k}\,\mathcal{D}_{2k}^{-2k}\,(-1)^k \sum_{\sigma\in S_{2k}} c(e_{\sigma(1)})\cdots c(e_{\sigma(2k)})\, d\mu_{\sigma(1)}\wedge\ldots\wedge d\mu_{\sigma(2k)} \\ &= (2k)!\,4^{-k}\,\mathcal{D}_{2k}^{-2k}\, i^k\,\gamma\, d\mu_1\wedge\ldots\wedge d\mu_{2k}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \big(\mathcal{P}-\frac{1}{2}\big)(d\mathcal{P})^{2k} &= \frac{-(2k)!\,i^k}{2^{2k+1}}\mathcal{Q}^{-1}\mathcal{D}_{2k}^{-2k}\gamma\big(D+c(\mu)\big)\gamma\, d\mu_1\wedge\ldots\wedge d\mu_{2k} \\ &= \frac{-(2k)!\,i^k}{2^{2k+1}}\mathcal{Q}^{-2k-1}\big(D-c(\mu)\big)\, d\mu_1\wedge\ldots\wedge d\mu_{2k}, \end{aligned} \tag{3.19}$$

and finally

$$\begin{aligned} &\overline{\mathrm{TR}}\Big(\big(\mathcal{P}-\frac{1}{2}\big)(d\mathcal{P})^{2k}\Big) \\ &\quad = \frac{-(2k)!\,i^k}{2^{k+1}} \int_{\mathbb{R}^{2k}} \mathrm{TR}\,\big(D(D^2+|\mu|^2)^{-(2k+1)/2}\big)\, d\mu \\ &\quad = -\frac{1}{2}(2\pi i)^k k!\,\eta(D). \end{aligned} \tag{3.20}$$

Here we have used (3.15) and the fact that the standard Clifford representation has rank 2^k.

Summing up we have proved the following even analogue of [19, Prop. 6.6].

Proposition 3.2. *Let D be an invertible first order self-adjoint elliptic differential operator. Let $\mathcal{D}_{2k}(\mu) := \gamma\,\big(D\otimes I_{\mathbb{C}^{2^k}} + c(\mu)\big), \mu\in\mathbb{R}^{2k}$, be the $2k$-fold suspension defined in Eq.* (3.16) *and let $\mathcal{P}$ be the idempotent defined in*

Eq. (3.17). *Then the η-invariant of D satisfies*

$$\eta(D) \;=\; -\frac{2}{(2\pi i)^k k!}\overline{\mathrm{TR}}\Big(\big(\mathcal{P}-\frac{1}{2}\big)(d\mathcal{P})^{2k}\Big).$$

Remark 3.3. This identity justifies promoting the above expression to a definition: the *higher (even) η-invariant* η_{2k} is defined on projections $\mathcal{P} \in \mathrm{CL}^0(M,E;\mathbb{R}^{2k})$ by

$$\eta_{2k}(\mathcal{P}) := -\frac{2}{(2\pi i)^k k!}\overline{\mathrm{TR}}\Big(\big(\mathcal{P}-\frac{1}{2}\big)(d\mathcal{P})^{2k}\Big). \tag{3.21}$$

Finally, we record the even analogue of Theorem 3.1.

Theorem 3.4. *Let $D_s : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ be a smooth family of elliptic first order self-adjoint differential operators on the vector bundle E such that D_0 and D_1 are invertible. Let $\mathcal{D}_s := \gamma\left(D_s \otimes I_{\mathbb{C}^{2^k}} + c(\mu)\right), \mu \in \mathbb{R}^{2k}$, be the $2k$-fold suspension defined in Eq.* (3.16). *Furthermore, let $\mathcal{P}_s \in \mathrm{CL}^1(M, E \otimes \mathbb{C}^{2^k};\mathbb{R}^{2k})$, $s \in [0,1]$, be a smooth family of almost idempotents with endpoints*

$$\mathcal{P}_j = \frac{1}{2}\big(I - (\mathcal{D}_j^2)^{-1/2}\mathcal{D}_j\big), \quad j = 0,1$$

and whose symbols satisfy

$$\sigma(\mathcal{P}_s) = \frac{1}{2}\big(I - \sigma(\mathcal{D}_s^2)^{-1/2}\sigma(\mathcal{D}_s)\big).$$

Then the even divisor flow of the family of almost idempotents $(\mathcal{P}_s)_{s\in[0,1]}$ coincides with the spectral flow of $(D_s)_{s\in[0,1]}$:

$$\mathrm{DF}\left((\mathcal{P}_s)_{0\le s\le 1}\right) = \mathrm{SF}\left((D_s)_{0\le s\le 1}\right). \tag{3.22}$$

Proof. We first prove the existence of a family of almost idempotents $(\mathcal{P}_s)_{0\le s\le 1}$ with the stated properties. Start with the smooth family

$$p_s = \frac{1}{2}\big(I - \sigma(\mathcal{D}_s^2)^{-1/2}\sigma(\mathcal{D}_s)\big) \tag{3.23}$$

of projections in $\mathrm{C}\mathcal{S}^0_{2k}$. In view of Eq. (2.13) and Eq. (2.14) there is a smooth lift $\widetilde{\mathcal{P}}_s \in \mathrm{CL}^1(M, E\otimes\mathbb{C}^{2^k};\mathbb{R}^{2k})$ with $\sigma(\widetilde{\mathcal{P}}_s) = p_s$. To adjust the endpoints we put

$$\mathcal{P}_s := \widetilde{\mathcal{P}}_s + s(\mathcal{P}_1 - \widetilde{\mathcal{P}}_1) + (1-s)(\mathcal{P}_0 - \widetilde{\mathcal{P}}_0), \tag{3.24}$$

which has all the desired properties.

Alternatively and even more concretely one can obtain $\mathcal{P}_s$ by modifying the construction of $\mathcal{Q}_s$ as follows: choose an even smooth function $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ with compact support such that $\phi(0) = 1$ and put

$$\mathcal{Q}_s := (\mathcal{D}_s^2 + \phi(\mu)\phi(D_s^2))^{1/2}. \tag{3.25}$$

$\mathcal{Q}_s$ is invertible and Eq. (3.17) now yields a smooth family of almost idempotents $\widetilde{\mathcal{P}}_s$ with $\sigma(\mathcal{P}_s) = p_s$. The endpoints are adjusted as before in Eq. (3.24).

Once such a family of almost idempotents is chosen, the proof of the statement is completely analogous to that of Theorem 3.1. The only difference is that for the variation of the even parametric η–invariant we cannot refer to [19] but have to use the transgression formula Eq. (1.37) and Eqs. (1.45), (1.46). More precisely, assume for the moment that D_s is invertible and let $(\varphi_{2k}, \psi_{2k+1})$ be the character of the cycle $C^{2k}_{\rm reg}$ (cf. Eq. (2.12)). Then from Proposition 3.2 and Eq. (1.45) we infer

$$\begin{aligned}\widetilde{\eta}(D_s) &= -\frac{1}{(2\pi i)^k k!}\overline{\rm TR}\Big(\big(\mathcal{P}_s - \frac{1}{2}\big)(d\mathcal{P}_s)^{2k}\Big),\\ &= \frac{(-1)^{k+1}}{(2\pi i)^k}\langle\varphi_{2k}, {\rm ch}_\bullet(\mathcal{P}_s)\rangle\end{aligned} \tag{3.26}$$

and hence Eq. (1.37), Proposition 1.10 and Eq. (1.46) yield

$$\begin{aligned}&\frac{d}{ds}\big(\widetilde{\eta}(D_s) \bmod \mathbb{Z}\big)\\ &= \frac{(-1)^{k+1}}{(2\pi i)^k}\langle\varphi_{2k}, (b+B)\,\not{\rm c}{\rm h}_\bullet(\mathcal{P}_s, (2\mathcal{P}_s - 1)\partial_s\mathcal{P}_s))\rangle\\ &= \frac{(-1)^{k+1}}{(2\pi i)^k}\langle\sigma^*\psi_{2k-1}, \not{\rm c}{\rm h}_\bullet(\mathcal{P}_s, (2\mathcal{P}_s - 1)\partial_s\mathcal{P}_s))\rangle\\ &= \frac{-1}{(2\pi i)^k k!}\widetilde{\rm TR}\Big(\sigma(2\mathcal{P}_s - 1)\sigma(\partial_s\mathcal{P}_s)(d\sigma(\mathcal{P}_s))^{2k-1}\Big).\end{aligned} \tag{3.27}$$

Thus we have established the analogue of Eq. (3.9) in the even case. As in the proof of Theorem 3.1 one now shows that the equality between the first and the last term in Eq. (3.27) holds for elliptic families D_s. Then inserting Eqs. (3.26) and (3.27) into the formula (1.44) for the divisor flow gives the claim. □

References

[1] Atiyah, M. F., Patodi, V. K., and Singer, I. M.: *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[2] Blackadar, B.: *K-Theory of Operator Algebras*, Springer Verlag, New York, 1986.

[3] Connes, A.: *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbb{R}$, Adv. Math. **39** (1981), 31–55.

[4] Connes, A.: *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360.

[5] Connes, A.: *Noncommutative Geometry*, Academic Press, 1994.

[6] Connes, A. and Moscovici, H.: *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990), no. 3, 345–388.

[7] Cuntz, J. and Quillen, D.: *Excision in bivariant periodic cyclic cohomology.* Invent. Math. **127** (1997), no. 1, 67–98.

[8] Elliott, G. A., Natsume, T., and Nest, R.: *Cyclic cohomology for one–parameter smooth crossed products*, Acta Math. **160** (1988), 285–305.

[9] Getzler, E.: *The odd Chern character in cyclic homology and the spectral flow.* Topology **32** (1993), no. 3, 489–507.

[10] Gilkey, P.: *Invariance theory, the heat equation, and the Atiyah–Singer index theorem*, Publish or Perish, Wilmington, DE, 1984.

[11] Gorokhovsky, A.: *Characters of cycles, equivariant characteristic classes and Fredholm modules*, Comm. Math. Phys. **208** (1999), 1–23.

[12] Gramsch, B.: *Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren*, Math. Ann. **209** (1984), 27–71.

[13] Grigis, A. and Sjøstrand, J.: *Microlocal analysis for differential operators*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, 1994.

[14] Higson, N. and Roe, J.: *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[15] Hörmander, L.: *Fourier integral operators. I*, Acta Math. **127** (1971), 79–183.

[16] Karoubi, M.: *K-theory – An introduction*, Grundlehren der mathematischen Wissenschaften, vol. 226, Springer–Verlag, Berlin–Heidelberg–New York, 1978.

[17] Kirk, P. and Lesch, M: *The η-invariant, Maslov index, and spectral flow for Dirac-type operators on manifolds with boundary*, Forum Math. **16** (2004), 553–629, math.DG/0012123.

[18] Lesch, M.: *On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols*, Ann. Global Anal. Geom. **17** (1999), 151–187.

[19] Lesch, M., and Pflaum, M.J. : *Traces on algebras of parameter dependent pseudodifferential operators and the eta-invariant*, Trans. Amer. Math. Soc. **352** (2000), no. 11, 4911–4936.

[20] Loday, J.L.: *Cyclic Homology.* Grundlehren der mathematischen Wissenschaften, vol. 301, Springer–Verlag, Berlin–Heidelberg–New York, 1992.

[21] Melrose, R.B.: *The eta invariant and families of pseudodifferential operators.* Math. Res. Lett. **2** (1995), no. 5, 541–561.

[22] Moscovici, H. and Wu, F.: *Index theory without symbols*, In: C^*-algebras: 1943–1993 (San Antonio, TX, 1993), pp. 304–351, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[23] Moroianu, S.: *K-theory of suspended pseudo-differential operators.* K-Theory **28** (2003), 167–181.

[24] Pflaum, M.J.: *The normal symbol on Riemannian manifolds*, The New York Journal of Mathematics **4** (1998), 95–123.

[25] Schweitzer, L. B.: *A short proof that $M_n(A)$ is local if A is local and Fréchet*, Intern. J. Math. **3** (1992), 581–589.

[26] Shubin, M. A.: *Pseudodifferential operators and spectral theory*, Springer–Verlag, Berlin–Heidelberg–New York, 1980.

[27] Swan, R.G.: *Topological examples of projective modules*, Trans. Amer. Math. Soc. **230** (1977), 201–234.

[28] Widom, H.: *A Complete Symbol Calculus for Pseudodifferential Operators*, Bull. Sci. Math. (2) **104** (1980), 19–63.

[29] Wodzicki, M.: *Local invariants of spectral asymmetry.* Invent. Math. **75** (1984), no. 1, 143–177.

[30] Wodzicki, M.: *Excision in cyclic homology and in rational algebraic K-theory.* Ann. of Math. (2) **129** (1989), no. 3, 591–639.

Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany
E-mail address: lesch@math.uni-bonn.de
URL: http://www.math.uni-bonn.de/people/lesch

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA
E-mail address: henri@math.ohio-state.edu

Fachbereich Mathematik, Goethe-Universität, 60054 Frankfurt/Main, Germany
E-mail address: pflaum@math.uni-frankfurt.de
URL: http://www.math.uni-frankfurt.de/∼pflaum