A HOMOLOGICAL APPROACH TO SINGULAR REDUCTION IN DEFORMATION QUANTIZATION

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Dedicated to Jean-Paul Brasselet on the occasion of his 60th birthday

Abstract. We use the method of homological quantum reduction to construct a deformation quantization on singular symplectic quotients in the situation, where the coefficients of the moment map define a complete intersection. Several examples are discussed, among others one where the singularity type is worse than an orbifold singularity.

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1. Introduction

In hamiltonian mechanics, reducing the number of degrees of freedom of a hamiltonian system by exploiting its symmetry is a standard method to determine the dynamics of the system. Within the language of symplectic geometry, regular reduction has been introduced independently by Meyer and Marsden/Weinstein and is usually called Marsden–Weinstein reduction. In [11] and, subsequently, in [3] it was shown that Marsden-Weinstein reduction has an analog in deformation quantization (see [7] for an overview on deformation quantization) in case the hamiltonian group action satisfies certain regularity conditions. This quantum reduction was used to obtain differentiable star products on regular symplectic quotient spaces. The general approach followed in [3] is known as the BRST-method and goes back to works of Batalin, Fradkin and Vilkoviski (for an overview and references on classical homological reduction see [24]).

In the following, we will see that the above method, suitably modified, works also for cases of *singular* reduction, where the singular behavior of the moment map is "not too bad". This will yield continuous star products on the corresponding singular quotient spaces. Let us be more specific about the premises to be made. We will consider a hamiltonian action of a connected and compact Lie group G acting on a symplectic manifold M with equivariant moment map $J : M \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$ of G. Let $Z := J^{-1}(0)$ be the zero set of J, it will be also called *constraint surface*. Due to the equivariance of J, the constraint surface is an invariant subset. Let us denote by $I(Z) \subset \mathcal{C}^\infty(M)$ the vanishing ideal of Z. We will assume that the moment map satisfies the following conditions:

a) the components of J generate $I(Z)$ (generating hypothesis),
b) the Koszul complex on J in the ring $\mathcal{C}^\infty(M)$ is acyclic (cf. Section 3).

Substantial work has been done in [1] in order to understand the generating hypothesis. Using local normal coordinates for the moment map this issue is reduced to a problem in algebraic geometry (cf. also Section 2). Note that the generating hypothesis puts severe restrictions on the geometry of Z: it implies that $I(Z)$ is a Poisson subalgebra. Using Dirac's terminology we say: Z is *first class*. If the Koszul complex is acyclic, one also says J is a *complete intersection* (see e.g. [4]). Misleadingly, the physicist's denotation is here: "J is irreducible". The question whether a variety is (locally) a complete intersection is fundamental in commutative algebra, but there the most interesting techniques to determine that rely on the assumption that the base ring is *noetherian*, as opposed to the ring of smooth functions on a manifold which is the base ring in our considerations. So we have to find alternatives and attack this problem directly by providing a simple crititerion for J to be a complete intersection (cf. Theorem 3.1). The proof may be interesting in its own right.

If zero is a singular value of the moment map, the constraint surface Z is not a smooth manifold, but, according to [23], a stratified space. A continuous function f on Z is said to be *smooth* if there is a smooth function $F \in \mathcal{C}^\infty(M)$ such that $f = F_{|Z}$. The algebra of smooth functions $\mathcal{C}^\infty(Z)$ is isomorphic to $\mathcal{C}^\infty(M)/I(Z)$. It is naturally a Fréchet algebra, since it is the quotient of a Fréchet algebra by a closed ideal. In [23] Sjamaar and Lerman could show that the orbit space of Z under the action of G is a stratified symplectic space. The Poisson algebra of smooth functions on it is naturally isomorphic to the Poisson algebra $\mathcal{C}^\infty(Z)^G/I(Z)^G$. Since Z is first class, $\mathcal{C}^\infty(Z)^{\mathfrak{g}}$ carries a canonical Poisson structure, which is referred to as the *Dirac reduced algebra*. Since G is compact and connected, these Poisson algebras are isomorphic. If the conditions a) and b) above are true, this Poisson algebra is identified with the zeroth cohomology of the classical BRST-algebra (cf. Section 4).

According to [3], it is relatively easy to find a formal deformation of the classical BRST-algebra into a differential graded associative algebra such that the cohomology is essentially unchanged (see Section 5 and 6), and thus yielding a deformation of the classical reduced Poisson structure. In [3] some efforts have been made to provide explicit formulas for contracting homotopies of the Koszul resolution, which have certain technical properties. Using these formulas and techniques from homological perturbation theory, it was shown that, in the regular case, a differentiable reduced star product can be found. Here we use the extension theorem and the division theorem of [2] to provide *continuous* contracting homotopies that satisfy similar technical assumptions.

In this way, we obtain the main result of this article. Given a hamiltonian action of a compact connected Lie Group on a symplectic manifold such that the moment map satisfies conditions a) and b) above, then there exists a *continuous* formal deformation of the Dirac reduced algebra, i.e. a continuous star product on the singular reduced space (see Corollary 6.4). Since it is clear, that a situation, where both conditions a) and b) are true, is rather special, we start the discussion by giving some examples (cf. Section 2). Needless to say, this will show that the theory does not reduce to the regular situation. But, more importantly, there are examples where the reduced spaces are not orbifolds, but genuine stratified symplectic spaces. To the authors knowledge, this is the first known instance of such a space admitting a deformation quantization. Homological reduction therefore provides a construction method for formal deformation quantizations which works for more general singular symplectic spaces than the Fedosov type construction introduced in [19] for orbifolds.

We have included an appendix providing basic notions of homological perturbation theory and two variants of the well known basic perturbation lemma (see e.g. [16]), which are less universal but fit our purposes. The perturbation lemma A.1 is also implicit in Fedosov's construction [10].

Througout this paper we shall use the following conventions. Unless otherwise stated, all complexes are *cochain* complexes in the category of $\mathbb{K}$-vector spaces, $\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$. The shift $V[j]$ of a graded vector space $V = \oplus_i V^i$ is defined by $V[j]^i := V^{i+j}$. If not said otherwise, maps of graded vector spaces are of degree zero. Concerning symplectic structure, moment maps, star products etc. we adopt the conventions of [3]. The formal parameter $\nu = i\lambda$ stands for $i\hbar$.

Acknowledgements. The authors would like to thank Markus Hunziker for stimulationg discussions and drawing our attention to important references concerning commuting varieties. We thank Paweł Domański for explaining the notion of a split, and Richard Cushman, Marc Gotay, Nolan Wallach and Patrick Erdelt for helpful advice. M.P. and H.-C.H. gratefully acknowledge support by Deutsche Forschungsgemeinschaft. H.-C.H. also acknowledges a travel stipend by Hermann Willkomm-Stiftung.

2. Examples

Before we start to explain the general machinery let us provide some examples of hamiltonian G-spaces, which satisfy the generating and the complete intersection hypothesis. In general, it is not at all a trivial matter to check, whether the generating hypothesis is true. The following is based on results of the seminal article [1]. We begin the discussion with the most simple case, where G is a torus.

2.1. **Hamiltonian torus actions.** In [1] it was proven that for a moment map $J : M \to \mathfrak{g}^*$ of a torus action to generate the vanishing ideal $I(Z)$, $Z = J^{-1}(0)$, it is necessary and sufficient, that the following *nonpositivity condition* applies: for all $\xi \in \mathfrak{g}$ and $z \in Z$ one has either

a) $J(\xi) = 0$ in a neighborhood $U \subset M$ of z, or
b) in every neighborhood $U \subset M$ of z the function $J(\xi)$ takes strictly positive as well as strictly negative values.

This nonpositivity condition and Theorem 3.1 make it easy to provide first nontrivial examples.

2.1.1. *Zero Angular Momentum for m particles in* $\mathbb{R}^2$. We consider the system of m particles in $\mathbb{R}^2$ with zero total angular momentum (see e.g. [17, Section 5] and [14, Section 6]). More precisely, the phase space is $M := (T^*\mathbb{R}^2)^m$ and we let $SO(2,\mathbb{R}) \cong S^1$ act on it by lifting the diagonal action, i.e.,

$$\begin{array}{rcl} SO(2) \times M & \to & M \\ (g, (\boldsymbol{q}_1, \boldsymbol{p}^1, \dots, \boldsymbol{q}_m, \boldsymbol{p}^m)) & \mapsto & (g\boldsymbol{q}_1, g\boldsymbol{p}^1, \dots, g\boldsymbol{q}_m, g\boldsymbol{p}^m), \end{array}$$

where $\boldsymbol{q}_i = (q_i^1, q_i^2)^t$ and $\boldsymbol{p}^i = (p_1^i, p_2^i)^t$ for $i = 1, \dots, m$. The moment map $J : M \to \mathfrak{so}(2) = \mathbb{R}$ is given by $J(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^m q_i^1 p_2^i - q_i^2 p_1^i$. In [17] the reduced space is described as a branched double cover of the closure of a certain coadjoint orbit of $\mathfrak{sp}(m, \mathbb{R})$. The moment map J obviously satisfies the nonpositivity condition above. Since $Z = J^{-1}(0)$ is of codimension 1, this implies that the Koszul complex (cf. Section 3) is a resolution of $\mathcal{C}^\infty(Z)$.

2.1.2. *An S^1-action with a worse-than-orbifold quotient.* The following example is taken from [6, p.125]. Consider the S^1-action on $\mathbb{C}^4$, endowed with symplectic form $\omega = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$, given by $\mathrm{e}^{i\vartheta} \cdot (z_1, z_2, z_3, z_4) := (\mathrm{e}^{\mathrm{i}\vartheta} z_1, \mathrm{e}^{\mathrm{i}\vartheta} z_2, \mathrm{e}^{-\mathrm{i}\vartheta} z_3, \mathrm{e}^{-\mathrm{i}\vartheta} z_4)$. The moment map for the action is

$$J(z_1, z_2, z_3, z_4) = \frac{1}{2}(|z_3|^2 + |z_4|^2 - |z_1|^2 - |z_2|^2).$$

The constraint surface Z is the real cone $C(S^3 \times S^3)$, and by a topological argument (see [6]), the reduced space $C(S^3 \times_{S^1} S^3)$ can not be an orbifold. Since J clearly satisfies the nonpositivity condition above, it generates the vanishing ideal $I(Z)$. Again, we conclude that the Koszul complex is a resolution of $\mathcal{C}^\infty(Z)$.

2.1.3. *A T^2-action on* $\mathbb{C}^4$. We consider example 7.7 from [1]. The action is given by $T^2 \times \mathbb{C}^4 \to \mathbb{C}^4$, $((\vartheta_1, \vartheta_2), (z_1, z_2, z_3, z_4)) \mapsto (\mathrm{e}^{\mathrm{i}(\alpha\vartheta_1 + \beta\vartheta_2)} z_1, \mathrm{e}^{-\mathrm{i}\vartheta_2} z_2, \mathrm{e}^{\mathrm{i}\vartheta_1} z_3, \mathrm{e}^{-\mathrm{i}\vartheta_2} z_4)$ for $\alpha, \beta \in \mathbb{Z}$. A moment map for the action is $J : \mathbb{C}^4 \to \mathbb{R}^2$, $J(z_1, z_2, z_3, z_4) := \frac{1}{2}(-\alpha|z_1|^2 - |z_3|^2, -\beta|z_1|^2 + |z_2|^2 - |z_4|^2)$. J satisfies the nonpositivity condition for $\alpha < 0$. An elementary calculation gives that also condition b) of Theorem 3.1 is true. Consequently, the corresponding Koszul complex is a resolution of the space of smooth functions on $Z := J^{-1}(0)$.

2.2. **Hamiltonian actions of nonabelian Lie groups.** As the nonpositivity condition, in the case of nonabelian group actions, is only *necessary* for the ideal $I(Z) \subset \mathcal{C}^\infty(M)$ to be generated by $J_1, \dots, J_\ell$, the reasoning here is usually more intricate. In [1] it was proven that the latter is the case iff in every normal coordinate system the ideal I generated by the moment map in the real polynomial ring $\mathbb{R}[x^1, \dots, x^{2n}]$ is *real* in the sense of real algebraic geometry (cf. [1, Theorem 6.3]). Recall that an ideal I in $\mathbb{R}[x^1, \dots, x^m]$ is real, if it coincides with its *real radical*

$$\sqrt[\mathbb{R}]{I} :=$$

$$\Big\{ f \in \mathbb{R}[x^1, \dots, x^m] \mid f^{2i} + \sum_{j=0}^k g_j^2 \in I \text{ for some } i \text{ and } g_1, \dots, g_k \in \mathbb{R}[x^1, \dots, x^m] \Big\}.$$

In [1] we find the following criterion for such an ideal to be real.

Theorem 2.1. *Let I be an ideal in $\mathbb{R}[x^1, \dots, x^m]$. Then I is real, if and only if the following two conditions hold:*

a) $I_{\mathbb{C}} := I \otimes_{\mathbb{R}} \mathbb{C}$ *is radical in* $\mathbb{C}[x^1, \dots, x^m]$*, and*

b) *for every irreducible component* $W \subset \mathbb{C}^m$ *of the (complex) locus of* $I_{\mathbb{C}}$

$$\dim_{\mathbb{R}}(W \cap \mathbb{R}^m) = dim_{\mathbb{C}}(W).$$

In other words, in order to know whether the ideal I is real, it is enough to gain detailed insight into the complex algebraic geometry behind the scene (e.g. knowing the primary decomposition of $I_{\mathbb{C}}$). Regardless the fact that the varieties in question are cones, there is no straightforward way to provide this information. A basic example, which one is tempted to consider is zero angular momentum of one particle in dimension n. Since the components of the moment map can be written as the 2×2-minors of a $2 \times n$-matrix, the ideal $I_{\mathbb{C}}$ is prime, and the complex locus is of dimension $n+1$ by a theorem of Hochster [13]. It follows easily, that the ideal I is real. Unfortunately, this example is not a complete intersection for $n \geq 3$. The only class of nonabelian examples, which the authors are aware of, where generating and complete intersection hypothesis are true at the same time, is the following.

2.2.1. *Commuting Varieties.* Let S the space of symmetric $n \times n$-matrices with real entries. We let $SO(n)$ act on S by conjugation and we lift this action to an action of $SO(n)$ on the cotangent bundle $T^*S = S \times S$. This action is hamiltonian with the moment map

$$\begin{aligned} J : S \times S &\rightarrow \wedge^2\mathbb{R}^n = \mathfrak{so}(n)^* \\ (Q,P) &\mapsto [Q,P], \end{aligned}$$

where we have identified $\mathfrak{so}(n)^*$ with the space $\wedge^2\mathbb{R}^n$ of antisymmetric $n \times n$-matrices. The complex locus $Z_{\mathbb{C}}$ defined by the these $\frac{1}{2}n(n-1)$ quadratic equations is an instance of what is called a *commuting variety*. In [5] it was shown that $Z_{\mathbb{C}}$ is irreducible of codimension $\frac{1}{2}n(n-1)$, and the ideal generated by the coefficients of J in the complex polynomial ring is prime. Let $S_{\mathrm{reg}} \subset S$ be the open subset of symmetric matrices with pairwise distinct eigenvalues. Since the action of $SO(n)$ on T^*S_{reg} is locally free, it follows that $Z \cap T^*S_{\mathrm{reg}}$ is of codimension $\frac{1}{2}n(n-1)$ likewise. As a consequence of Theorem 2.1, the components of J generate the vanishing ideal $I(Z)$ in $\mathcal{C}^\infty(T^*S)$. It is easy to see, that T_zJ is surjective for $z \in Z \cap T^*S_{\mathrm{reg}}$. By Theorem 3.1 below, the Koszul complex is a resolution of the space of smooth function on Z. Using invariant theory, the reduced space was identified in [17] as the quotient $(\mathbb{R}^n \times \mathbb{R}^n)/S_n$, the symmetric group S_n acting diagonally. Note that the results of [5] have been generalized to moment maps of the isotropy representations of symmetric spaces of maximal rank [18].

3. Koszul resolution

Given a smooth map $J : M \rightarrow \mathbb{R}^\ell =: V^*$ we consider the Koszul holomogical complex of the sequence of ring elements $J_1, \ldots, J_\ell \in \mathcal{C}^\infty(M)$, but we will view it later artificially as a cochain complex. In other words, we define the space of (co)chains to be $K^i := K_{-i}(M,J) := S^i_{\mathcal{C}^\infty(M)}(V[1])$, i.e. the free (super)symmetric $\mathcal{C}^\infty(M)$-algebra generated by the graded vector space $V[1]$, where we consider V to be concentrated in degree zero. $K_\bullet$ may also be viewed as the space of sections of the trivial vector bundle over M with fibre $\wedge^\bullet V$. Denoting by $e^1, \ldots, e^\ell$ the canonical bases of the dual space V of $V^* = \mathbb{R}^\ell$, we define the Koszul differential $\partial := \sum_a J_a i(e^a)$, where the $i(e^a)$ are the derivations extending the dual pairing. We will say, in accordance with [4], that $J_1, \ldots, J_\ell \in \mathcal{C}^\infty(M)$ is a *complete intersection*, if the homology of the Koszul complex vanishes in degree $\neq 0$.

Now we would like to have a simple geometric criterion for J to be a complete intersection. We achieve this goal only after knowing that J generates the vanishing ideal (which is sometimes difficult to decide).

Theorem 3.1. *Let M be an analytic manifold and $J : M \to \mathbb{R}^\ell$ an analytic map, such that the following conditions are true*

a) $(J_1, \ldots, J_\ell)$ *generate the vanishing ideal of* $Z := J^{-1}(0)$ *in* $\mathcal{C}^\infty(M)$,
b) *the regular stratum* $Z_r := \{z \in Z | T_z J$ *is surjective*$\}$ *is dense in* $Z := J^{-1}(0)$.

Then the Koszul complex $K := K(M, J)$ *is acyclic and* $H_0 = \mathcal{C}^\infty(Z)$.

Proof. We will show that the Koszul complex $K(\mathcal{C}^\omega_x(M), J)$ is acyclic for the ring $\mathcal{C}^\omega_x(M)$ of germs in x of real analytic functions. Then it will follow that the Koszul complex $K(\mathcal{C}^\infty(M), J)$ is acyclic, since the ring of germs of smooth functions $\mathcal{C}^\infty_x(M)$ is flat over $\mathcal{C}^\omega_x(M)$ (see [25, p.118]), and the sheaf of smooth functions on M is fine. Since $\mathcal{C}^\omega_x(M)$ is noetherian, Krull's intersection theorem says that $\cap_{r\geq 0} I^r_x = 0$, where I_x is the ideal of germs of analytic functions vanishing on Z. According to [4, A X.160], it is therefore sufficient to show that $H_1(\mathcal{C}^\omega_x(M), J) = 0$. Note that since J generates the vanishing ideal of Z in $\mathcal{C}^\infty(M)$, it also generates the vanishing ideal of Z in $\mathcal{C}^\omega_x(M)$. This can easily be seen using M. Artin's approximation theorem (see e.g.[21]). Suppose $f = \sum_a f^a\, e_a \in K_1$ is a cycle, i.e. $\partial f = \sum_a J_a f^a = 0$. Since the restriction to Z of the Jacobi matrix $D(\sum_a J_a f^a)$ vanishes, we conclude (using condition b)) that $f^a_{|Z} = 0$ for all $a = 1, \ldots, \ell$. Since J generates the vanishing ideal, we find an $\ell \times \ell$-matrix $F = (F^{ab})$ with smooth (resp. analytic) entries such that $f^a = \sum_b F^{ab} J_b$. It remains to be shown, that this matrix can be choosen to be *antisymmetric*. We have to distinguish two cases. If $x \notin Z$, the claim is obvious, since then one can take for example $F^{ab} := (\sum_a J_a^2)^{-1}(J_b f^a - J_a f^b)$. So let us consider the other case $x \in Z$. We then introduce some formalism to avoid tedious symmetrization arguments. Let E denote the free $k := \mathcal{C}^\omega_x(M)$-module on ℓ generators, and consider the Koszul-type complex $SE \otimes \wedge E$. Generators of the symmetric part will be denoted by $\mu_1, \ldots, \mu_\ell$, generators of the Grassmann part by $e_1, \ldots, e_\ell$, respectively. We have two derivations $\delta := \sum_a e_a \wedge \frac{\partial}{\partial \mu_a} : S^n E \otimes \wedge^m E \to S^{n-1} E \otimes \wedge^{m+1} E$, and $\delta^* := \sum_a \mu_a i(e^a) : S^n E \otimes \wedge^m E \to S^{n+1} E \otimes \wedge^{m-1} E$. They satisfy the well known identities: $\delta^2 = 0$, $(\delta^*)^2 = 0$ and $\delta\delta^* + \delta^*\delta = (m+n)\,\mathrm{id}$. Furthermore, we introduce the two commuting derivations $i_J := \sum_a J_a i(e^a)$ and $d_J = \sum_a J_a \frac{\partial}{\partial \mu_a}$. They obey the identities $i_J^2 = 0$, $[i_J, \delta] = d_J$, $[d_J, \delta^*] = i_J$ and $[i_J, \delta^*] = 0 = [d_J, \delta]$. We interprete the cycle f above as being in $E \otimes k$ and the matrix F as a member of $E \otimes E$. We already know that $d_J f = 0$ implies $f = i_J F$. This argument may be generalized as follows: if $a \in S^n E \otimes k$ obeys $d_J^n a = 0$, then there is an $A \in S^n E \otimes E$ such that $a = i_J A$. The proof is easily provided by taking all n-fold partial derivatives of $d_J^n a = 0$, evaluating the result on Z and using conditions a) and b). We now claim that there is a sequence of $F_{(n)} \in S^{n+1} E \otimes E$, $n \geq 0$, such that $F = F_{(0)}$, $\delta^* F_{(n)} = (n+2) i_J F_{(n+1)}$ and

$$f = d_J^n i_J F_{(n)} + i_J \delta^* \underbrace{\Big(\sum_{i=0}^{n-1} \frac{1}{i+2}\, d_J^i \delta F_{(i)} \Big)}_{=:B_{n-1}} \qquad \text{for all } n \geq 1. \tag{3.1}$$

We prove this by induction. Setting $B_{-1} := 0$, we may start the induction with $n = 0$, where nothing has to be done. Suppose now, that the claim is true for $F_{(0)}, \dots, F_{(n)}$. We obtain $f = \frac{1}{n+2} d_J^n i_J\big(\delta\delta^* F_{(n)} + \delta^*\delta F_{(n)}\big) + i_J\delta^* B_{n-1} = \frac{1}{n+2} d_J^{n+1}\delta^* F_{(n)} + i_J\delta^* B_n$, where we made use of the relations $[d_J^n i_J, \delta^*] = 0$ and $[d_J^n i_J, \delta] = d_J^{n+1}$. Since $0 = d_J f = d_J^{n+2}\delta^* F_{(n)}$, we find an $F_{(n+1)}$ such that $\frac{1}{n+2}\delta^* F_{(n)} = i_J F_{(n+1)}$, and the claim is proven. Finally, we want to take the limit of equation (3.1) as n goes to ∞. For this limit to make sense, we have to change the ring to the ring of formal power series. Let us denote this change of rings by $\hat{}: \mathcal{C}^\omega_x(M) \to \mathbb{K}[[x^1, \dots, x^n]]$. Since by Krull's intersection theorem $\cap_{r\geq 0} \hat{I}^r = 0$ ($\hat{I}$ the ideal generated by $\hat{J}_1, \dots, \hat{J}_\ell$), we obtain a formal solution of the problem: $\hat{f} = i_{\hat{J}}\delta^* B_\infty$, where $B_\infty := \sum_{i=0}^\infty \frac{1}{i+2}\, d_{\hat{J}}^i \delta \hat{F}_{(i)}$ is well defined since $\hat{I}$ contains the maximal ideal. Applying M. Artin's approximation theorem yields an analytic solution, and we are done. □

The above reasoning can be considered to be folklore, as the subtlety of finding an *antisymmetric* source term is often swept under the rug in semirigorous arguments. The next theorem though is a consequence of rather deep analytic results. The problem of splitting the Koszul resolution in the context of Fréchet spaces was also addressed in [8] from a different perspective.

Theorem 3.2. *Let M be a smooth manifold, $J : M \to \mathbb{R}^\ell$ be smooth map such that around every $m \in M$ there is a local chart in which J is real analytic. Moreover, assume that the Koszul complex $K = K(M, J)$ is a resolution of $\mathcal{C}^\infty(Z)$, $Z = J^{-1}(0)$. Then there are a prolongation map $\mathrm{prol} : \mathcal{C}^\infty(Z) \to \mathcal{C}^\infty(M)$ and contracting homotopies $h_i : K_i \to K_{i+1}$, $i \geq 0$, which are continuous in the respective Fréchet topologies, such that*

$$\big(\mathcal{C}^\infty(Z), 0\big) \underset{\mathrm{prol}}{\overset{\mathrm{res}}{\leftrightarrows}} \big(K, \partial\big), h \qquad (3.2)$$

is a contraction, i.e. res *and* prol *are chain maps and* $\mathrm{res}\ \mathrm{prol} = \mathrm{id}$ *and* $\mathrm{id} - \mathrm{prol}\ \mathrm{res} = \partial h + h\partial$. *If necessary, these can be adjusted in such a way, that the side conditions (see Appendix A)* $h_0\ \mathrm{prol} = 0$ *and* $h_{i+1}\, h_i = 0$ *are fulfilled. If, moreover, a compact Lie group G acts smoothly on M, G is represented on $\mathbb{R}^\ell$ and $J : M \to \mathbb{R}^\ell$ is equivariant, then* prol *and h can additionally be chosen to be equivariant.*

Proof. A closed subset $X \subset \mathbb{R}^n$ is defined to have the *extension property*, if there is a continuous linear map $\lambda : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(\mathbb{R}^n)$, such that $\mathrm{res}\ \lambda = \mathrm{id}$. The extension theorem of E. Bierstone and G. W. Schwarz, [2, Theorem 0.2.1] says that Nash subanalytic sets (and hence closed analytic sets) have the extension property. Using a partition of unity, we get a continuous linear map $\lambda : \mathcal{C}^\infty(Z) \to \mathcal{C}^\infty(M)$, such that $\mathrm{res}\ \lambda = \mathrm{id}$. In the same reference, one finds a "division theorem" (Theorem 0.1.3.), which says that for a matrix $\varphi \in \mathcal{C}^\omega(\mathbb{R}^n)^{r,s}$ of analytic functions the image of $\varphi : \mathcal{C}^\infty(\mathbb{R}^n)^s \to \mathcal{C}^\infty(\mathbb{R}^n)^r$ is closed, and there is a continuous split $\sigma : \mathrm{im}\,\varphi \to \mathcal{C}^\infty(\mathbb{R}^n)^s$ such that $\varphi\,\sigma = \mathrm{id}$. Using a partition of unity, we conclude that there are linear continuous splits $\sigma_i : \mathrm{im}\,\partial_{i+1} \to K_{i+1}$ for the Koszul differentials $\partial_{i+1} : K_{i+1} \to K_i$ for $i \geq 0$, i.e. $\partial_{i+1}\,\sigma_i = \mathrm{id}$. We observe that $\mathrm{im}\,\lambda \oplus \mathrm{im}\,\partial_1 = K_0$, since for every $x \in K_0$ the difference $x - \lambda\ \mathrm{res}\, x$ is a boundary due to exactness and the sum is apparantly direct. Similarly, we show that $\mathrm{im}\,\sigma_i \oplus \mathrm{im}\,\partial_{i+2} = K_{i+1}$ for $i \geq 0$. The next step is to show that $\mathrm{im}\,\sigma_i$ is a *closed* subspace of K_0. Therefor we assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{im}\,\partial_{i+1}$ such that $\sigma_i(x_n)$ converges to $y \in K_{i+1}$. Then $x_n = \partial_{i+1}\sigma_i(x_n)$ converges to $\partial_{i+1}y$, since ∂_{i+1} is continuous. Since $\partial_{i+1}y$ is in the domain of σ_i, we obtain that $\sigma_i(x_n)$ converges to $\sigma_i\partial_{i+1}y = y \in \mathrm{im}\,\sigma_i$. Similarly, we have that $\mathrm{im}\,\lambda$ is a closed subspace of K_0. Altogether, it is feasible to extend σ_i to a linear continuous map $K_i \to K_{i+1}$ (cf. [20, p.133]). If necessary, λ and σ_i can be made equivariant by averaging over G, since res and ∂ are equivariant. We observe that we have $\lambda\,\mathrm{res}_{|\mathrm{im}\lambda} = \mathrm{id}$ and $\lambda\,\mathrm{res}_{|\mathrm{im}\partial_1} = 0$ and analogous equations in higher degrees. We now replace λ by $\mathrm{prol} := \lambda - \partial_1\sigma_0\lambda$ and σ_i by $h_i := \sigma_i - \partial_{i+2}\sigma_{i+1}\sigma_i$ for $i \geq 0$. These maps share all of the above mentioned properties with λ and σ_i. Additionally, we have $\partial_1 h_{0|\mathrm{im(prol)}} = 0$ and $\partial_{i+2}h_{i+1|\mathrm{im}(h_i)} = 0$ for $i \geq 0$. This concludes the construction of (3.2). The side conditions can be achieved by algebraic manipulations (see Appendix A). Note that these modifications do not ruin the equivariance. □

A crucial property of the Koszul resolution is that it is a differential graded commutative algebra. In the present context, where the constraint functions are the components of a moment map, it has the following extra feature. The Lie algebra $\mathfrak{g}$ acts on it by even derivations, extending the actions on $\mathfrak{g}$ and on $\mathcal{C}^\infty(M)$.

4. Classical homological reduction

The BRST-algebra is defined to be $\mathscr{A} := S_{\mathcal{C}^\infty(M)}\big(\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]\big)$, i.e. the free graded commutative $\mathcal{C}^\infty(M)$-algebra generated by $\mathfrak{g}$ (of degree -1) and $\mathfrak{g}^*$ (of degree 1). We adopt the usual convention to call the elements of $\mathfrak{g}^*$ and $\mathfrak{g}$ *ghosts* and *antighosts*, respectively. We will frequently refer to a basis $e_1, \dots, e_\ell$ and $e^1, \dots, e^\ell$ of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (we will use latin indices: $a, b, \dots$). There is an even graded Poisson bracket on $\mathscr{A}$ extending that on M, which is uniquely defined by the requirements $\{\alpha, x\} = \langle\alpha, x\rangle$ and $\{f, x\} = 0 = \{f, \alpha\}$ for all $x \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$ and $f \in \mathcal{C}^\infty(M)$. With the Lie bracket and the moment map we build an element $\theta := -\frac{1}{4}\sum_{a,b,c} f^c_{ab}e^a e^b e_c + \sum_a J_a e^a \in \mathscr{A}^1$, where the f^c_{ab} are the structure constants of $\mathfrak{g}$. An easy calculation yields $\{\theta, \theta\} = 0$, hence $\mathscr{D} := \{\theta, ?\}$ is a differential. Summing up, we obtain a differential graded Poisson algebra $(\mathscr{A}, \{,\}, \mathscr{D} = \{\theta, ?\})$, we call θ the *BRST-charge* and $\mathscr{D}$ the *classical BRST-differential*.

Closer examination shows that $\mathscr{D} = \delta + 2\partial$ is a linear combination of two supercommuting differentials. Here, δ is the codifferential of the Lie algebra cohomology corresponding to the $\mathfrak{g}$-module $S_{\mathcal{C}^\infty(M)}(\mathfrak{g}[1])$, this representation will be denoted by L, and $\partial = \sum_a J_a i^a$ is the extension of the Koszul differential. We view $\mathscr{D}$ as a perturbation (see Appendix A) of the acyclic differential ∂.

We extend the restriction map res to a map $\mathrm{res} : \mathscr{A} \to S_{\mathcal{C}^\infty(Z)}(\mathfrak{g}^*[-1])$ by setting it zero for all terms containing antighosts and restricting the coefficients. In the same fashion, we extend prol to a map $S_{\mathcal{C}^\infty(Z)}(\mathfrak{g}^*[-1]) \to \mathscr{A}$ extending the coefficients.

Since the moment map J is G-equivariant, G acts on $Z = J^{-1}(0)$. Hence $\mathcal{C}^\infty(Z)$ is a $\mathfrak{g}$-module, this representation will be denoted by L^z. Note that $\mathrm{L}^z_X = \mathrm{res}\ \mathrm{L}_X\ \mathrm{prol}$ for all $X \in \mathfrak{g}$. We identify $S_{\mathcal{C}^\infty(Z)}(\mathfrak{g}^*[-1])$ with the space of cochains of Lie algebra cohomology $C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)$. Let us denote $d : C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big) \to C^{\bullet+1}\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)$ the codifferential of Lie algebra cohomology coresponding to L^z. Since res is a morphism of $\mathfrak{g}$-modules we obtain $d\ \mathrm{res} = \mathrm{res}\ \delta$.

Theorem 4.1. *There are $\mathbb{K}$-linear maps $\Phi : C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big) \to \mathscr{A}^\bullet$ and $H : \mathscr{A}^\bullet \to \mathscr{A}^{\bullet-1}$ which are continuous in the respective Fréchet topologies such that*

$$\Big(C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big), d\Big) \underset{\Phi}{\overset{\text{res}}{\leftrightarrows}} (\mathscr{A}^\bullet, \mathscr{D}), H \tag{4.1}$$

is a contraction.

Proof. Apply lemma A.1 to the perturbation $\mathscr{D}_\nu$ of 2∂. Explicitly, we get $H := \frac{1}{2}h\sum_{j=0}^{\ell}(-\frac{1}{2})^j(h\delta + \delta h)^j$ and $\Phi = \mathrm{prol} - H(\delta\ \mathrm{prol} - \mathrm{prol}\ d)$, which are obviously Fréchet continuous. Note that from $h\,\mathrm{prol} = 0$ and $h^2 = 0$ it follows that $H\Phi = 0$ and $H^2 = 0$. If prol is chosen to be equivariant, then the expression for Φ simplifies to $\Phi = \mathrm{prol}$. In the same way one gets $H = \frac{1}{2}h$, if h is equivariant. □

Corollary 4.2. *There is a graded Poisson structure on $\mathrm{H}^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)$. If $[a], [b]$ are the cohomology classes of $a, b \in C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)$, then the bracket is given given by $\{[a], [b]\} := [\mathrm{res}\{\Phi(a), \Phi(b)\}]$. The restriction of this bracket to $\mathrm{H}^0\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big) = \mathcal{C}^\infty(Z)^\mathfrak{g}$ coincides with the Dirac reduced Poisson structure.*

5. The quantum BRST-algebra

In this section we will introduce the quantum BRST algebra, which is $\mathbb{K}[[\nu]]$-dg algebra $(\mathscr{A}^\bullet[[\nu]], *, \mathscr{D}_\nu)$ deforming the classical dg Poisson algebra $(\mathscr{A}^\bullet, \{,\}, \mathscr{D})$. The exposition parallels that of [3]. In order to define a graded product $*$ on $\mathscr{A}[[\nu]]$, we use on one hand a Clifford multiplication $x \cdot y := \mu\big(\mathrm{e}^{-2\nu\sum_a i^a \otimes i_a}(x \otimes y)\big)$ for $x, y \in S_\mathbb{K}(\mathfrak{g}[1] \oplus \mathfrak{g}[-1])$. Here μ denotes the supercommutative multiplication, i^a and i_a are the left derivations extending the dual pairing with e^a and e_a, respectively and $\otimes$ denotes the graded tensor product. On the other hand, we will need a star product $\star$ on M, which is compatible with the $\mathfrak{g}$-action in the following sense

$$\mathbb{J}(X) \star \mathbb{J}(Y) - \mathbb{J}(Y) \star \mathbb{J}(X) = \nu\mathbb{J}([X, Y]) \text{ for all } X, Y \in \mathfrak{g}, \tag{5.1}$$

where $\mathbb{J} = J + \sum_{i\geq 1}\nu^i J_{(i)} \in \mathscr{A}^1[[\nu]]$ is a deformation of the moment map J. In other words, $\star$ is *quantum covariant* for the *quantum moment map* $\mathbb{J}$. For $f, g \in \mathcal{C}^\infty(M)$ and $x, y \in S\big(\mathfrak{g}[1] \oplus \mathfrak{g}^*[-1]\big)$ we define $(fx) * (gy) := (f \star g)\,(x \cdot y)$. Note that $*$ is graded. The next step is to quantize the BRST-charge. Luckily, we are done with (see e.g. [15])

$$\theta_\nu := -\frac{1}{4}\sum_{a,b,c} f_{ab}^c\, e^a e^b e_c + \sum\nolimits_a \mathbb{J}_a\, e^a + \frac{1}{2}\nu\sum_a f_{ab}^b e^a \ \in \mathscr{A}^1[[\nu]],$$

since a straightforward calculation yields $\theta_\nu * \theta_\nu = 0$. We define the *quantum BRST differential* to be $\mathscr{D}_\nu := \frac{1}{\nu}\,\mathrm{ad}_*(\theta_\nu)$.

Before we take a closer look, at $\mathscr{D}_\nu$ let us introduce some terminology. We define the (superdifferential) operators $\delta_\nu, \mathscr{R}, q, u : \mathscr{A}^\bullet \to \mathscr{A}^{\bullet+1}$,

$$\begin{aligned}
\delta_\nu(f) &:= -\frac{1}{2}\sum_{a,b,c} f_{ab}^c\, e^a e^b\, i_c(f) + \sum_{a,b,c} f_{ab}^c\, e^a e_c\, i_b(f) + \sum_a e^a\, \frac{1}{\nu}[\mathbb{J}_a, f]_*, \\
\mathscr{R}(f) &:= \sum_a i^a\, f * \mathbb{J}_a, && \text{“right multiplication”} \\
q(f) &:= -\frac{1}{2}\sum_{a,b,c} f_{ab}^c\, e_c\, i^a i^b(f), && \text{“quadratic ...”} \\
u(f) &:= \sum_{a,b} f_{ab}^b\, i^a(f), && \text{“unimodular term”}
\end{aligned}$$

for $f \in \mathscr{A}$. Note that δ_ν is the coboundary operator of Lie algebra cohomology corresponding to the representation

$$\begin{aligned}
\mathbb{L}_X : S_{\mathcal{C}^\infty(M)}(\mathfrak{g}[1])[[\nu]] &\to S_{\mathcal{C}^\infty(M)}(\mathfrak{g}[1])[[\nu]], \\
af &\mapsto (\mathrm{ad}_X(a))f + a\,\nu^{-1}(\mathbb{J}(X) \star f - f \star \mathbb{J}(X)),
\end{aligned} \tag{5.2}$$

where $X \in \mathfrak{g}$, $a \in S_{\mathbb{K}}(\mathfrak{g}[1])$ and $f \in \mathcal{C}^\infty(M)[[\nu]]$. Finally, we set

$$\partial_\nu := \mathscr{R} + \nu\Big(\frac{1}{2}u - q\Big).$$

This operator will be called the *deformed* or *quantum Koszul differential*. Note that ∂_ν is a homomorphism of $\mathcal{C}^\infty(M)[[\nu]]$-left-modules. As a side remark, ∂_ν may also be interpreted as a differential of Lie algebra homology of a certain representaion of $\mathfrak{g}$. This point of view was adopted in [22].

Theorem 5.1. *The quantum BRST differential $\mathscr{D}_\nu = \delta_\nu + 2\partial_\nu$ is a linear combination of two supercommuting differentials δ_ν and ∂_ν.*

Proof. Straightforward calculation. □

6. Quantum reduction

The main idea, which we follow in order to compute the quantum BRST cohomology (i.e. the cohomology of $(\mathscr{A}[[\nu]], \mathscr{D}_\nu)$), is to provide a deformed version of the contraction (4.1). This will be done by applying Lemma A.2 to the contraction (3.2) for the perturbation ∂_ν of ∂ and then applying Lemma A.1 for the perturbation $\mathscr{D}_\nu$ of $2\partial_\nu$. We will also need to examine a deformed version of the representation L^z of $\mathfrak{g}$ on $\mathcal{C}^\infty(Z)$.

Proposition 6.1. *If we choose h_0 such that $h_0\,\mathrm{prol} = 0$, then there are deformations of the restriction map $\mathrm{res}_\nu = \mathrm{res} + \sum_{i\geq 1} \nu^i\ \mathrm{res}_i : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(Z)[[\nu]]$ and of the contracting homotopies $h_{\nu i} = h_i + \sum_{j\geq 1} \nu^j\ h_i^j : K_i[[\nu]] \to K_{i+1}[[\nu]]$, which are a formal power series of Fréchet continuous maps and such that*

$$\big(\mathcal{C}^\infty(Z)[[\nu]], 0\big) \underset{\mathrm{prol}}{\overset{\mathrm{res}_\nu}{\leftrightarrows}} \big(K[[\nu]], \partial_\nu\big), h_\nu \tag{6.1}$$

is a contraction with $h_{\nu 0}\ \mathrm{prol} = 0$. Explicitly, we have

$$\mathrm{res}_\nu := \mathrm{res}\ (\mathrm{id} + (\partial_{\nu 1} - \partial_1)h_0)^{-1}.$$

If we choose h to be $\mathfrak{g}$-equivariant, the same is true for h_ν.

Proof. Apply lemma A.2 to the perturbation ∂_ν of ∂. $\square$

We now define the quantized representation $\mathbb{L}^z$ of $\mathfrak{g}$ on $\mathcal{C}^\infty(Z)[[\nu]]$ by setting

$$\mathbb{L}^z_X := \mathrm{res}_\nu \ \mathbb{L}_X \ \mathrm{prol} \quad \text{for } X \in \mathfrak{g}.$$

That this is in fact a representation, follows easily from the observation $\mathbb{L}_X \partial_\nu - \partial_\nu \mathbb{L}_X = 0$ for all $X \in \mathfrak{g}$ (this is a consequence of Theorem 5.1), and from h_ν being a contracting homotopy. In the same fashion as in Section 4, we define $d_\nu : C^\bullet(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]) \to C^{\bullet+1}(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]])$ to be the differential of Lie algebra cohomology of the representation $\mathbb{L}^z$, i.e. $d_\nu \ \mathrm{res}_\nu = \mathrm{res}_\nu \ \delta_\nu$. In the same manner, we extend res_ν and h_ν as in Section 4 to maps $\mathrm{res}_\nu : \mathscr{A} \to C\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big)$ and $h_\nu : \mathscr{A}^\bullet[[\nu]] \to \mathscr{A}^{\bullet-1}[[\nu]]$.

Theorem 6.2. *There are $\mathbb{K}[[\nu]]$-linear maps $\Phi_\nu : C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big) \to \mathscr{A}^\bullet[[\nu]]$ and $H_\nu : \mathscr{A}^\bullet \to \mathscr{A}^{\bullet-1}[[\nu]]$, which are series of Fréchet continuous maps such that*

$$\Big(C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big), d_\nu\Big) \underset{\Phi_\nu}{\overset{\mathrm{res}_\nu}{\leftrightarrows}} (\mathscr{A}^\bullet[[\nu]], \mathscr{D}_\nu), H_\nu$$

is a contraction.

Proof. Since the requisite condition $\mathrm{res}_\nu \ h_\nu = 0$ is obviously fulfilled, we apply Lemma A.1 to the perturbation $\mathscr{D}_\nu$ of $2\partial_\nu$. Explicitly, this means that $H_\nu := \frac{1}{2} h_\nu \sum_{j=0}^{\ell} (-\frac{1}{2})^j (h_\nu \delta_\nu + \delta_\nu h_\nu)^j$ and $\Phi_\nu = \mathrm{prol} - H_\nu(\delta_\nu \ \mathrm{prol} - \mathrm{prol} \ d_\nu)$, which are obviously series of Fréchet continuous maps. Note that from $h_0 \,\mathrm{prol} = 0$ and $h^2 = 0$, we get $H_\nu \Phi_\nu = 0$ and $H_\nu^2 = 0$. If prol is chosen to be equivariant, then the expression for Φ simplifies to $\Phi_\nu = \mathrm{prol}$. If h and (hence h_ν) is equivariant, then it follows that $H_\nu = \frac{1}{2} h_\nu$. $\square$

We use this contraction to transfer the associative algebra structure from $\mathscr{A}[[\nu]]$ to the Lie algebra cohomology $\mathrm{H}^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big)$ of the representation $\mathbb{L}^z$ by setting

$$[a] * [b] := \big[\mathrm{res}_\nu \big(\Phi_\nu(a) * \Phi_\nu(b)\big)\big] \tag{6.2}$$

where $[a], [b]$ denote the cohomology classes of $a, b \in C^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big)$. But in fact that is *not exactly*, what we want to accomplish. The primary obstacle on the way to the main result, Corollary 6.4, is that, in general, we have $\mathrm{H}^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]]\big) \neq \mathrm{H}^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)[[\nu]]$. An example where this phenomenon occurs was given in [3, section 7]. One way out is to sharpen the compatibility condition (5.1). We require, that $\mathbb{J} = J$ and

$$J(X) \star f - f \star J(X) = \nu\{J(X), f\} \quad \text{for all } X \in \mathfrak{g}, f \in \mathcal{C}^\infty(M).$$

This property is also referred to as *strong invariance* of the star product $\star$ with respect to the Lie algebra action. For proper group actions a strongly invariant star product can always be found (see [9]). Of course, now the representations $\mathbb{L}$ and L coincide and we get $\delta = \delta_\nu$. But with some mild restrictions on the contracting homotopy h of the Koszul resolution we also have the following.

Lemma 6.3. *If h_0 is $\mathfrak{g}$-equivariant and $h_0 \,\mathrm{prol} = 0$, then $\mathbb{L}^z = \mathrm{L}^z$.*

Proof. For $X \in \mathfrak{g}$ we have $\mathbb{L}^z_X = \mathrm{res}_\nu \ \mathrm{L}_X \ \mathrm{prol} = \mathrm{res}\,(\mathrm{id} + (\partial_{\nu 1} - \partial_1) h_0)^{-1} \mathrm{L}_X \ \mathrm{prol}$. Since L_X commutes with $\partial_{\nu 1}$, ∂_1 and h_0 the last expression can be written as $\mathrm{res}\ \mathrm{L}_X(\mathrm{id} + (\partial_{\nu 1} - \partial_1) h_0)^{-1} \,\mathrm{prol} = \mathrm{res}\ \mathrm{L}_X \ \mathrm{prol}$. $\square$

Corollary 6.4. *With the assumptions made above the product defined by equation (6.2) makes* $\mathrm{H}^\bullet\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)[[\nu]]$ *into a graded associative algebra. For the subalgebra* $\mathrm{H}^0\big(\mathfrak{g}, \mathcal{C}^\infty(Z)\big)[[\nu]] = \big(\mathcal{C}^\infty(Z)\big)^{\mathfrak{g}}[[\nu]]$ *this formula simplifies to*

$$f * g := \mathrm{res}_\nu \left(\mathrm{prol}(f) * \mathrm{prol}(g)\right) \quad \text{for } f, g \in \big(\mathcal{C}^\infty(Z)\big)^{\mathfrak{g}}. \tag{6.3}$$

Since $\big(\mathcal{C}^\infty(Z)\big)^{\mathfrak{g}}[[\nu]]$ *is* $\mathbb{K}[[\nu]]$*-linearly isomorphic to the algebra of smooth functions on the symplectic stratified space* M_{red}*, we obtain an associative product on* $\mathcal{C}^\infty(M_{\mathsf{red}})[[\nu]]$ *which gives rise to a* continuous *Hochschild cochain.*

Proof. It remains to show (6.3). Let us denote by $\mathscr{A}^+$ the kernel of the augmentation map $\mathscr{A} \to \mathcal{C}^\infty(M)$. Equation (6.3) follows from the fact that $(\mathscr{A}^+ \cap \mathscr{A}^0)[[\nu]]$ is a two-sided ideal in $\mathscr{A}^0[[\nu]]$. □

Finally, if $\mathrm{H}^1(\mathfrak{g}, C^\infty(Z))$ vanish, it is possible to find a topologically linear isomorphism between the spaces of invariants for the classical and the deformed representation.

Corollary 6.5. *Let G be a compact, connected semisimple Lie group acting on the Poisson manifold M in a Hamiltonian fashion. Assume that the equivariant moment map J satisfies the generating and the complete intersection hypothesis. Then for a star product $*$ on M with quantum moment map $\mathbb{J}$ there is an invertible sequence of continuous maps*

$$S = \sum_{i \geq 0} \nu^i \, S_i : \mathrm{H}^0(\mathfrak{g}, \mathcal{C}^\infty(Z))[[\nu]] = \mathcal{C}^\infty(Z)^{\mathfrak{g}}[[\nu]] \to \mathrm{H}^0(\mathfrak{g}, \mathcal{C}^\infty(Z)[[\nu]])$$

such that the formula

$$f \star g := S^{-1}\big(S(f) * S(g)\big) = S^{-1}\Big(\mathrm{res}_\nu \left(\Phi_\nu(S(f)) * \Phi_\nu(S(g))\right)\Big)$$

defines a continuous formal deformation of the Poisson algebra $\mathcal{C}^\infty(Z)^{\mathfrak{g}}$ *into an associative algebra.*

Proof. According to Viktor L. Ginzburg (see [12, Theorem 2.13]) we have for any compact, connected Lie group G with a smooth representation on a Fréchet space W an isomorphism $\mathrm{H}^\bullet(\mathfrak{g}, W) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathbb{K}) \otimes W^{\mathfrak{g}}$. In particular, this implies that if $\mathfrak{g}$ is semisimple, the first and the second cohomology groups of the $\mathfrak{g}$-module $\mathcal{C}^\infty(Z)$ vanish. Note, that, since G is compact, the space of invariants $\mathcal{C}^\infty(Z)^{\mathfrak{g}} \subset \mathcal{C}^\infty(Z)$ has a closed complement. This can be taken to be the kernel of the averaging projection. Using these observations it is straight forward to construct S by a standard inductive argument (see e.g. [3, p.140]). □

Appendix A. Two perturbation lemmata

We consider (cochain) complexes in an additive $\mathbb{K}$-linear category $\mathscr{C}$ (e.g. the category of Fréchet spaces). A *contraction* in $\mathscr{C}$ consists of the following data

$$(X, d_X) \underset{i}{\overset{p}{\leftrightarrows}} (Y, d_Y), h_Y, \tag{A.1}$$

where i and p are chain maps between the chain complexes (X, d_X) and (Y, d_Y), $h_Y : Y \to Y[-1]$ is a morphism, and we have $pi = \mathrm{id}_X$, $d_Y h_Y + h_Y d_Y = \mathrm{id}_Y - ip$. The contraction is said to satisfy the *side conditions* (sc1–3), if moreover, $h_Y^2 = 0$, $h_Y i = 0$ and $p h_Y = 0$ are true. It was observed in [16], that in order to fulfill (sc2) and (sc3), one can replace h_Y by $h'_Y := (d_Y h_Y + h_Y d_Y)\, h_Y\, (d_Y h_Y + h_Y d_Y)$. If one wants to have in addition (sc1) to be satisfied, one may relapce h'_Y by $h''_Y := h'_Y d_Y h'_Y$. Let $C := \mathrm{Cone}(p)$ be the mapping cone of p, i.e. $C = X[1] \oplus Y$ is the complex with differential $d_C(x,y) := (d_X x + (-1)^{|y|} py, d_Y y)$. The homology of C is trivial, because $h_C(x,y) := (0, h_Y y + (-1)^{|x|} ix)$ is a contracting homotopy, i.e. $d_C h_C + h_C d_C = \mathrm{id}_C$, if (sc3) is true.

Let us now assume that the objects X and Y carry complete descending filtrations and the structure maps are filtration preserving. Moreover, pretend that we have found a *perturbation* $D_Y = d_Y + t_Y$ of d_Y, i.e. $D_Y^2 = 0$ and $t_Y : Y \to Y[1]$, called the *initiator*, has the property that $t_Y h_Y + h_Y t_Y$ raises the filtration. Since, in general, $t_X := p t_Y i$ needs not to be a perturbation of d_X, we impose that as an extra condition: we *assume* that $D_X = d_X + t_X$ is a differential. Setting $t_C := (t_X, t_Y)$, we will get a perturbation $D_C := d_C + t_C$ of d_C, if we have in addition $t_X p = p t_Y$ (this will imply that $(d_X + t_X)^2 = 0$). Then an easy calculation yields that $H_C := h_C (D_C h_C + h_C D_C)^{-1} = h_C (\mathrm{id}_C + t_C h_C + h_C t_C)^{-1}$ is well defined and satisfies $D_C H_C + H_C D_C = \mathrm{id}_C$. Defining the morphism $I : X \to Y$, $H_C(x,0) =: (0, (-1)^{|x|} Ix)$ and the homotopy $H_Y : Y \to Y[-1]$, $H_C(0,y) =: (0, H_Y y)$ we get the following

Lemma A.1 (*Perturbation Lemma – Version 1*)**.** *If the contraction (A.1) satisfies (sc3) and $D_Y = d_Y + t_Y$ is a perturbation of d_Y such that $t_X p = p t_Y$, then*

$$(X, D_X) \underset{I}{\overset{p}{\leftrightarrows}} (Y, D_Y), H_Y, \tag{A.2}$$

is a contraction fulfilling (sc3). Moreover, we have $H_Y = h_Y(\mathrm{id}_Y + t_Y h_Y + h_Y t_Y)^{-1}$ and $Ix = ix - H_Y(t_Y ix - i t_X x)$. If all side conditions are true for (A.1), then they are for (A.2), too.

Starting with the mapping cone $K = \mathrm{Cone}(i)$, i.e. the complex $K = Y[1] \oplus X$ with the differential $d_K(y,x) = (d_Y y + (-1)^{|x|} ix, d_X x)$, we may give a version of the above argument arriving at a contraction with all data perturbed except i. More precisely, we have a homotopy $h_K(y,x) := (h_Y y, (-1)^{|y|} py)$, for which $d_K h_K + h_K d_K = \mathrm{id}_K$ follows from (sc2). Mimicking the above argument, we get a differential $D_K := d_K + t_K$ with $t_K := (t_Y, t_X)$, if $t_Y i = i t_X$ (this will imply $D_X^2 = 0$). Assuming (A.1) to satisfy (sc2), $H_K := h_K (D_K h_K + h_K D_K)^{-1}$ will become a contracting homotopy $D_K H_K + H_K D_K = \mathrm{id}_K$. Defining $P : Y \to X$ and $H'_Y : Y \to Y[-1]$ by $H_K(y,0) = H_K(y,x) =: (H'_Y y, (-1)^{|y|} Py)$ we get the following

Lemma A.2 (*Perturbation Lemma – Version 2*)**.** *If the contraction (A.1) satisfies (sc2) and $D_Y = d_Y + t_Y$ is a perturbation of d_Y such that $t_Y i = i t_X$, then*

$$(X, D_X) \underset{i}{\overset{P}{\leftrightarrows}} (Y, D_Y), H'_Y, \tag{A.3}$$

is a contraction fulfilling (sc2). Moreover, we have $H'_Y = h_Y(\mathrm{id}_Y + t_Y h_Y + h_Y t_Y)^{-1}$ and $P = p(\mathrm{id} + t_Y h_Y + h_Y t_Y)^{-1}$. If all side conditions are true for (A.1), then they are for (A.3), too.

References

1. Judith M. Arms, Mark J. Gotay, and George Jennings, *Geometric and algebraic reduction for singular momentum maps*, Adv. Math. **79** (1990), no. 1, 43–103.
2. Edward Bierstone and Gerald W. Schwarz, *Continuous linear division and extension of $\mathcal{C}^\infty$ functions*, Duke Math. J. **50** (1983), no. 1, 233–271.
3. Martin Bordemann, Hans-Christian Herbig, and Stefan Waldmann, *BRST cohomology and phase space reduction in deformation quantization*, Comm. Math. Phys. **210** (2000), no. 1, 107–144.
4. Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1980, Algèbre. Chapitre 10. Algèbre homologique.
5. J. P. Brennan, M. V. Pinto, and W. V. Vasconcelos, *The Jacobian module of a Lie algebra*, Trans. Amer. Math. Soc. **321** (1990), no. 1, 183–196.
6. Richard Cushman and Reyer Sjamaar, *On singular reduction of Hamiltonian spaces*, Symplectic geometry and mathematical physics (Aix-en-Provence, 1990) (P. Donato, C. Duval, J. Elhadad, and G. M. Tuynman, eds.), Progr. Math., vol. 99, Birkhäuser Boston, Boston, MA, 1991, pp. 114–128.
7. Giuseppe Dito and Daniel Sternheimer, *Deformation quantization: genesis, developments and metamorphoses*, Deformation quantization (Strasbourg, 2001), IRMA Lect. Math. Theor. Phys., vol. 1, de Gruyter, Berlin, 2002, pp. 9–54.
8. P. Domański and B. Jakubczyk, *Linear continuous division for exterior and interior products*, Proc. Amer. Math. Soc. **131** (2003), no. 10, 3163–3175 (electronic).
9. Boris Fedosov, *Deformation quantization and index theory*, Mathematical Topics, vol. 9, Akademie Verlag, Berlin, 1996.
10. Boris V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), no. 2, 213–238.
11. ______, *Non-abelian reduction in deformation quantization*, Lett. Math. Phys. **43** (1998), no. 2, 137–154.
12. Viktor L. Ginzburg, *Equivariant Poisson cohomology and a spectral sequence associated with a moment map*, Internat. J. Math. **10** (1999), no. 8, 977–1010.
13. Melvin Hochster, *Topics in the homological theory of modules over commutative rings*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975.
14. Johannes Huebschmann, *Lie-Rinehart algebras, descent, and quantization*, Galois theory, Hopf algebras, and semiabelian categories, Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 295–316.
15. Bertram Kostant and Shlomo Sternberg, *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, Ann. Physics **176** (1987), no. 1, 49–113.
16. Larry Lambe and Jim Stasheff, *Applications of perturbation theory to iterated fibrations*, Manuscripta Math. **58** (1987), no. 3, 363–376.
17. Eugene Lerman, Richard Montgomery, and Reyer Sjamaar, *Examples of singular reduction*, Symplectic geometry (Coventry, 1990) (Dietmar Salamon, ed.), London Math. Soc. Lecture Note Ser., vol. 192, Cambridge Univ. Press, Cambridge, 1993, pp. 127–155.
18. Dmitrii I. Panyushev, *The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties*, Compositio Math. **94** (1994), no. 2, 181–199.
19. Markus J. Pflaum, *On the deformation quantization of symplectic orbispaces*, Diff. Geometry and its Applications **19** (2003), 343–368.
20. Walter Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.
21. Jesús M. Ruiz, *The basic theory of power series*, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1993.
22. Alexey Sevostyanov, *Reduction of quantum systems with arbitrary first class constraints and Hecke algebras*, Comm. Math. Phys. **204** (1999), no. 1, 137–146.
23. Reyer Sjamaar and Eugene Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. (2) **134** (1991), no. 2, 375–422.
24. Jim Stasheff, *Homological reduction of constrained Poisson algebras*, J. Differential Geom. **45** (1997), no. 1, 221–240.

25. Jean-Claude Tougeron, *Idéaux de fonctions différentiables*, Springer-Verlag, Berlin, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 71.

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