MATH 6230 Differential Geometry 1
Homework Set 5

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Problem 1: Let M be a smooth manifold, $\varrho \in \Omega^k(M)$, and $\omega \in \Omega^l(M)$. Show that

$$i_X(\varrho \wedge \omega) = (i_X \varrho) \wedge \omega + (-1)^k \varrho \wedge (i_X \omega).$$

Problem 2: Prove the following proposition.
Let M be a differentiable manifold and $\iota_t : M \to M \times [0, \infty[$ with $t \in [0, \infty[$ the embedding $x \mapsto (x, t)$. Define the operator $K_t : \Omega^{l+1}(M \times [0, t]) \to \Omega^l(M)$ by

$$K_t(\omega)(y)\,(v_1, \cdots, v_l) = \int_0^t \omega(y, s) \left(\frac{\partial}{\partial s}, v_1, \cdots, v_l\right) ds,$$
$$\omega \in \Omega^{l+1}(M \times [0, t]),\ y \in M,\ v_1, \cdots, v_l \in T_y M.$$

Then K_t satisfies the equality $dK_t + K_t d = \iota_t^* - \iota_0^*$, hence for every smooth homotopy $H : M \times [0, t] \to M$ the relation

$$dK_t H^* + K_t H^* d = H_t^* - H_0^* \tag{1}$$

follows, where $H_s = H(\cdot, s)$ for $s \in [0, t]$.

Problem 3:

a) Derive from Problem 2 the Lemma of Poincaré, which says that for a ball $B \subset \mathbb{R}^n$ the de Rham complex $\Omega^\bullet(B)$ is exact.
Hint: Consider the homothety $H : B \times [0, 1] \to B$, $(t, v) \mapsto tv$.

b) Show that for every manifold M the sequence of sheaves

$$0 \to \underline{\mathbb{R}}_M \to \Omega^0_M \to \Omega^1_M \to \ldots \to \Omega^{\dim M}_M \to 0$$

is exact.
Hint: Exactness of a sequence of sheaves means that over each footpoint $p \in M$ the induced sequence of stalks is exact.

Problem 4: Prove that for a compact manifold M every vector field is integrable.