MATH 6230 Differential Geometry 1
Homework Set 4

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Problem 1: Let M be a (real) manifold of finite dimension and fix a point $p \in M$. Let $\mathcal{U}_p$ be the set of open neighborhoods of p in M and denote by $\mathcal{C}^\infty(\mathcal{U}_p)$ the set of smooth functions $f : U \to \mathbb{R}$, where U is an element of $\mathcal{U}_p$. Define an equivalence relation $\sim$ on $\mathcal{C}^\infty(\mathcal{U}_p)$ by calling two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ with $U, V \in \mathcal{U}_p$ *equivalent* if there exists an element $W \in \mathcal{U}_p$ such that $W \subset U \cap V$ and $f|W = g|W$. Denote by $\mathcal{C}^\infty_p$ the set of equivalence classes and call it the *stalk* of $\mathcal{C}^\infty(M)$ at p. Its elements are called *germs* of smooth functions at p. For $f \in \mathcal{C}^\infty(\mathcal{U}_p)$ the associated equivalence class is denoted by $[f]_p$.

(a) Construct a natural algebra structure on the stalk $\mathcal{C}^\infty_p$ reflecting addition and multiplication of smooth functions.

(b) Call a linear map $\delta : \mathcal{C}^\infty_p \to \mathbb{R}$ a *derivation* if

$$\delta([f]_p \cdot [g_p]) = f(p)\delta([g_p]) + g(p)\delta([f]_p) \quad \text{for all } [f]_p, [g]_p \in \mathcal{C}^\infty_p \ .$$

Denote the space of such derivations by $\mathrm{Der}(\mathcal{C}^\infty_p, \mathbb{R})$. Show that $\mathrm{Der}(\mathcal{C}^\infty_p, \mathbb{R})$ is a vector space.

(c) Show that the map

$$T_pM \to \mathrm{Der}(\mathcal{C}^\infty_p, \mathbb{R}), \quad \dot\gamma(0) \mapsto \left(\mathcal{C}^\infty_p \ni [f]_p \mapsto \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \in \mathbb{R} \right)$$

is well defined and a linear isomorphism.

Remark. (c) shows that the tangent space T_pM can be identified with the space of derivations $\mathrm{Der}(\mathcal{C}^\infty_p, \mathbb{R})$. This is the *algebraists definition* of the tangent space.

Problem 2: In the wikipedia article `https://en.wikipedia.org/wiki/Sheaf_(mathematics)` read the sections with the definition of presheaves, sheaves, on examples and on the stalks of a sheaf. Then define the presheaf $\mathcal{X}^\infty_M$ of smooth vector fields on a manifold M and the presheaf of k-forms Ω^k_M on M. Show that these presheaves are sheaves.

Problem 3: Verify that the space of smooth vector fields $\mathcal{X}^\infty(M)$ on a manifold M coincides naturally with the space of derivations $\mathrm{Der}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$.
Hint. First provide a precise definition of the space $\mathrm{Der}(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$ and then use Problems 1 and 2 to construct the desired isomorphism.

Problem 4: Let M be a manifold and E a $\mathcal{C}^\infty(M)$-module. Denote by $\mathcal{L}^k_{\mathcal{C}^\infty(M)}(E, \mathcal{C}^\infty(M))$ the space of k-fold $\mathcal{C}^\infty(M)$-multilinear maps from the k-fold cartesian product $E \times \ldots \times E$ to $\mathcal{C}^\infty(M)$ and by $\mathcal{A}^k_{\mathcal{C}^\infty(M)}(E, \mathcal{C}^\infty(M))$ the subspace of alternating multilinear maps. Show that there is a natural isomorphism

$$\Omega^k(M) \to \mathcal{A}^k_{\mathcal{C}^\infty(M)}(\mathcal{X}^\infty(M), \mathcal{C}^\infty(M)) \ .$$