MATH 6230 Differential Geometry 1
Homework Set 3

Spring 2022

Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 255, Telephone: 2-7717, e-mail: `markus.pflaum@colorado.edu`.

Problem 1: In this problem, $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ equipped with its canonical real manifold structure. Define $f : \mathbb{C}^2 \to \mathbb{C}$ by $f(w,z) = w^2 - z^3$, and let $V = f^{-1}(0)$.

(a) Show that f is a submersion at each $(w,z) \in \mathbb{C}^2 \setminus \{0,0\}$ and conclude from that that $V \setminus \{(0,0)\}$ is a submanifold of $\mathbb{C}^2$ of real dimension 2.

(b) Show that V intersects the unit sphere $S^3 \subset \mathbb{C}^2$ transversally which means that

$$T_pV + T_pS^3 = \mathbb{C}^2$$

for all $p \in S^3 \cap V$.
(Hint: Consider the path $\gamma \colon \mathbb{R}_{>0} \to V$ given by $\gamma(t) = (t^3 w, t^2 z)$.)

(c) Conclude from (b) that the intersection $S^3 \cap V$ is a 1-manifold K.
(Hint: Prove that the intersection of two submanifolds of a manifold M which intersect transversally is a submanifold and find a formula for the dimension of the intersection).

Problem 2: Consider the torus $S^1 \times S^1$. Let $\alpha \in \mathbb{R}$ and $\varphi_\alpha : \mathbb{R} \to S^1 \times S^1$ be the map

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t}).$$

For which α is the image of φ_α a submanifold of $S^1 \times S^1$, for which α is it not a submanifold.

Problem 3: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = x^3 + xy + y^3 + 1.$$

For which of the points $p = (0,0)$, $p = (\frac{1}{3}, \frac{1}{3})$ and $p = (-\frac{1}{3}, -\frac{1}{3})$ is the preimage $f^{-1}(f(p))$ a submanifold?

Problem 4: Let $U = \mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \binom{\cos\varphi_0}{\sin\varphi_0}$ with $\varphi_0 \in \mathbb{R}$. Consider corresponding polar coordinates $y : U \to \mathbb{R}_{>0} \times]\varphi_0 - 2\pi, \varphi_0[$, i.e. let $y^{-1}(r,\varphi) = r\binom{\cos\varphi}{\sin\varphi}$. Represent the vector fields $r\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \varphi}$ in cartesian coordinates.