MATH 6230 Differential Geometry 1
Homework Set 1

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Problem 1: Consider the gluing construction defined in class.

(a) Show that if the family $(U_i, x_i)_{i\in I}$ is countable, then the resulting topology on M is second countable.

(b) Show that M is Hausdorff, if and only if the image of the canonical diagonal embedding $U_i \cap U_j \hookrightarrow U_i \times U_j$ is closed for all $i, j \in I$. Hereby, the factors of $U_i \times U_j$ carry the unique topologies such that x_i and x_j are homeomorphisms.

Problem 2: Consider the general linear group $\mathsf{GL}(n, \Bbbk)$ with $\Bbbk = \mathbb{R}$ or $= \mathbb{C}$. Show that $\mathsf{GL}(n, \Bbbk)$ is an open subset of $\Bbbk^{n^2}$. Then prove that the following maps are smooth and compute their derivatives:

(a) the inversion $\mathsf{GL}(n, \Bbbk) \to \mathsf{GL}(n, \Bbbk)$, $g \mapsto g^{-1}$,

(b) the product map $\mathsf{GL}(n, \Bbbk) \times \mathsf{GL}(n, \Bbbk) \to \mathsf{GL}(n, \Bbbk)$, $(g, h) \mapsto gh$,

(c) conjugation $\mathsf{GL}(n, \Bbbk) \times \mathsf{GL}(n, \Bbbk) \to \mathsf{GL}(n, \Bbbk)$, $(g, h) \mapsto ghg^{-1}$.

Derive that $\mathsf{GL}(n, \Bbbk)$ is a Lie group.

Problem 3: Let M be a compact manifold of positive dimension. Show that there is a no global chart for M, but that M has a finite atlas.

Problem 4: Show that the n-dimensional torus $T^n = (S^1)^n$ is an n-dimensional smooth manifold. What is the minimum number of charts one needs to cover the n-torus?