Math 6220 Introduction to Topology 2
Homework Set 2

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By a semi-cosimplicial module (over a given ring R) one understands a sequence of R-modules M^k, $k \in \mathbb{N}$ together with R-module maps $d^i : M^k \to M^{k+1}$, $(0 \le i \le k+1)$ such that

$$d^j \, d^i = d^i \, d^{j-1} \text{ for } i < j.$$

Show that then $(M^\bullet, d)$ with $d = \sum_{i=0}^{k+1} (-1)^i d^i$ is a cochain complex. Its cohomology groups will be denoted by $H^k(M)$.

Singular cohomology. Let X be a topological space, and $\Bbbk$ a unital commutative ring. Put $S^k(X) := \operatorname{Hom}_{\mathbb{Z}}(S_k(X), \Bbbk)$, and let d^i be dual to the face map $\delta_i : S_k(X) \to S_{k-1}(X)$. Prove that one thus obtains a semi-cosimplicial module. The corresponding cohomology is called singular cohomology of X with coefficients on $\Bbbk$.

Čech cohomology. Let X be a topological space, and $\mathcal{U}$ an open covering of X. Denote for each k by $\mathcal{U}^{(k)}$ the set of all $(k+1)$-tupels $(U_0, \ldots, U_k)$ of elements of $\mathcal{U}$ such that $U_0 \cap \ldots \cap U_k \neq \emptyset$. For each $k \in \mathbb{N}$ let

$$\check{C}^k(X) := \prod_{(U_0,\ldots,U_k) \in \mathcal{U}^{(k)}} \underline{\mathbb{R}}(U_0 \cap \ldots \cap U_k),$$

where for an open set $U \subset X$ one denotes by $\underline{\mathbb{R}}(U)$ the space of locally constant functions from U to $\mathbb{R}$. Define now $\check{\partial}_i : \check{C}^k(X) \to \check{C}^{k+1}(X)$ by

$$\left(f_{(U_0,\ldots,U_k)}\right)_{(U_0,\ldots,U_k) \in \mathcal{U}^{(k)}} \mapsto \left(f_{(U_0,\ldots U_{i-1},U_{i+1},\ldots,U_{k+1})}\right)_{(U_0,\ldots,U_{k+1}) \in \mathcal{U}^{(k+1)}}.$$

Check that one thus obtains a semi-cosimplicial space; its cohomology is called the Čech cohomology of X with respect to the covering $\mathcal{U}$ and is denoted by $\check{H}^\bullet(X)$.

Alexander–Spanier cohomology. Let X be a topological space. For $k \in \mathbb{N}$ denote by $C^k(X)$ the space of all continuous functions f from X^{k+1} to $\mathbb{R}$. Define face maps $\delta^i : C^k(X) \to C^{k+1}(X)$ by the formula

$$\delta^i f(x_0, \ldots, x_{k+1}) = f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+1}), \quad f \in C^k(X).$$

Show that one thus obtains a semi-cosimplicial space. The resulting cochain complex $\big(C^\bullet(X), \delta\big)$ is acyclic. By passing to an appropriate quotient complex one obtains a complex which has nontrivial cohomology and contains the information one wants. To define that quotient complex consider the subcomplex $C_0^\bullet(X)$ of *locally vanishing* cochains. In degree k, this complex consists of all $f \in C^k(X)$ for which there exists an open covering $\mathcal{U}$ of X such that f vanishes on the neighborhood $\mathcal{U}^{(k+1)} = \bigcup_{U \in \mathcal{U}} U^{k+1}$. Show that δf is locally vanishing, if f is. The quotient complex $\big(C^k_{\mathrm{AS}}(X), \delta\big) := \big(C^\bullet(X)/C_0^\bullet(X), \delta\big)$ is called the *Alexander–Spanier complex* of X (with coefficients in $\mathbb{R}$), and its cohomology the *Alexander–Spanier cohomology* of X. One denotes the Alexander–Spanier cohomology of X by $H^\bullet_{\mathrm{AS}}(X)$.