Math 6220 Introduction to Topology 2
Homework Set 1

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Problem 1. For a given topological space X let $\check{X}$ denote the Stone-Čech compactification of X, i.e. the closure of the image of the canonical map

$$\iota : X \to [0,1]^{\mathcal{C}(X,[0,1])}, \quad x \to \big(f(x)\big)_{f \in \mathcal{C}(X,[0,1])},$$

where $[0,1]^{\mathcal{C}(X,[0,1])}$ carries the product topology. Show that the Stone-Čech compactification is a functor from the category of completely regular topological spaces to the category of compact topological spaces.

Problem 2. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between two categories. Show that F is an equivalence of categories, if and only if the following holds true:

- F is *full*, i.e. for any two objects X and Y of C, the map $\mathrm{Mor}_{\mathsf{C}}(X,Y) \to \mathrm{Mor}_{\mathsf{D}}(FX,FY)$ induced by F is surjective;
- F is *faithful*, i.e. for any two objects X and Y of C, the map $\mathrm{Mor}_{\mathsf{C}}(X,Y) \to \mathrm{Mor}_{\mathsf{D}}(FX,FY)$ induced by F injective; and
- F is *essentially surjective*, i.e. each object Z in D is isomorphic to an object of the form FX for some object X in C.

Problem 3. Show that

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 \\ \downarrow & & \downarrow \\ \mathbb{Z}/6 & \longrightarrow & \mathbb{Z}/12 \end{array}$$

is cocartesian in the category of abelian groups and that

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 \\ \downarrow & & \downarrow \\ \mathbb{Z}/6 & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) \end{array}$$

is cocartesian in the category of groups.

Problem 4. Recall that the *simplicial category* Simp consists of objects $\langle n \rangle$, where $n \in \mathbb{N}$ and $\langle n \rangle$ denotes the ordered set of integers $0 < 1 < \ldots < n$, and of morphisms given by non-decreasing maps $f : \langle n \rangle \to \langle m \rangle$. Define the *face* and *degenerecy maps* $\delta_{n,i} : \langle n-1 \rangle \to \langle n \rangle$ resp. $\sigma_{n,i} : \langle n+1 \rangle \to \langle n \rangle$ as follows, with $0 \leq i \leq n$:

$$\delta_{n,i}(l) = \begin{cases} l & \text{for } 0 \leq l < i, \\ l+1 & \text{for } i \leq l \leq n-1, \end{cases}$$

$$\sigma_{n,i}(l) = \begin{cases} l & \text{for } 0 \leq l \leq i, \\ l-1 & \text{for } i < l \leq n+1, \end{cases}$$

If by the context it is clear which maps are meant, we will often write δ_i for $\delta_{n,i}$ and σ_i for $\sigma_{n,i}$. Prove that the category Simp has the following properties:

(i) The only isomorphisms are the identity morphisms $\mathrm{id}_{\langle n\rangle}$.

(ii) The face and degeneracy maps satisfy the following commutation relations:

$$\begin{aligned}
\delta_{n+1,j}\,\delta_{n,i} &= \delta_{n+1,i}\,\delta_{n,j-1} \quad \text{for } 0 \le i < j \le n+1,\\
\sigma_{n-1,j}\,\sigma_{n,i} &= \sigma_{n-1,i}\,\sigma_{n,j+1} \quad \text{for } 0 \le i \le j \le n-1,\\
\sigma_{n,j}\,\delta_{n+1,i} &= \begin{cases} \delta_{n,i}\,\sigma_{n-1,j-1} & \text{for } 0 \le i < j \le n,\\ \mathrm{id}_{\langle n\rangle} & \text{for } i = j \text{ and } i = j+1,\\ \delta_{n,i-1}\,\sigma_{n-1,j} & \text{for } 1 \le j+1 < i \le n+1\ . \end{cases}
\end{aligned}$$

(iii) Every morphism $f : \langle n\rangle \to \langle m\rangle$ has a unique decomposition of the form

$$f = \delta_{i_r} \cdot \ldots \cdot \delta_{i_1} \cdot \sigma_{j_1} \cdot \ldots \cdot \sigma_{j_s}\ ,$$

where $i_1 < \ldots < i_r$, $j_1 < \ldots < j_s$ and $m = n - s + r$