Math 2002 Number Systems
Homework Set 8

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Problem 1: Call a sequence $(x_n)_{n\in\mathbb{R}}$ of real numbers convergent to a real number x if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|x - x_n| < \varepsilon \quad \text{for all } n \geq N \ .$$

Using that $\mathbb{R}$ is Dedekind complete show that every bounded and monotone increasing sequence $(x_n)_{n\in\mathbb{R}}$ converges to the supremum of the set $\{x_n \in \mathbb{R} \mid n \in \mathbb{N}\}$. (4P)

Hint: Recall that a sequence $(x_n)_{n\in\mathbb{R}}$ is called monotone increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

Note: A corresponding result holds for bounded monotone decreasing sequences.

Problem 2: Let $a > 0$ be a real number, fix $x_0 > 0$ and define $(x_n)_{n\in\mathbb{N}}$ recursively as follow:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \ .$$

Then show the following:

(a) $x_n > 0$ for all $n \in \mathbb{N}$. (2P)

(b) $x_n^2 \geq a$ for all $n \geq 1$. (2P)

(c) $x_{n+1} \leq x_n$ for all $n \geq 1$. (2P)

(d) Put $y_n = \frac{a}{x_n}$. Then $y^2 \leq a$ for all $n \geq 1$. (2P)

(e) $y_n \leq y_{n+1}$ for all $n \geq 1$. (2P)

(f) $y_n \leq x_n$ for all $n \geq 1$. (2P)

(g) The sequence $(x_n)_{n\in\mathbb{N}}$ converges. (2P)

(h) The limit $x = \lim_{n\to\infty} x_n$ satisfies $x^2 = a$. (2P)