Math 2002 Number Systems
Homework Set 7

Fall 2022

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Problem 1: Given $p \in \mathbb{N}$ with $p \geq 2$ define the relation $\sim_p$ of *congruence mod p* for integers as follows:

$$m \sim_p n \quad \text{if and only if there exists } k \in \mathbb{Z} \text{ such that } p \cdot k = m - n \ .$$

If m is congruent n mod p one also writes $m \equiv n \mod p$.

(a) Show that congruence mod p is an equivalence relation on $\mathbb{Z}$. Denote for each $m \in \mathbb{Z}$ by $\overline{m}$ its equivalence class and by $\mathbb{Z}/p\mathbb{Z}$ the set of equivalence classes. (2P)

(b) Verify that the following maps are well-defined:

$$+ : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z},\ (\overline{m}, \overline{n}) \mapsto \overline{m+n},$$
$$\cdot : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z},\ (\overline{m}, \overline{n}) \mapsto \overline{m \cdot n} \ .$$

(2P)

(c) Prove that for p a prime number the sets $\mathbb{Z}/p\mathbb{Z}$ together with the above maps $+$ and $\cdot$ and the elements $\overline{0}$ and $\overline{1}$ are fields. What is the cardinality of the field $\mathbb{Z}/p\mathbb{Z}$? (5P)

(d) Again under the assumption that p is prime show that there is no order relation on the field $\mathbb{Z}/p\mathbb{Z}$ turning it into an ordered field. (3P)

Problem 2: Let $\left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}}$ denote the set of all *Cauchy sequences* in $\mathbb{Q}$ that is the set of all sequences $(x_n)_{n \in \mathbb{N}}$, where $x_n \in \mathbb{Q}$ for $n \in \mathbb{N}$, such that for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m \geq N \ .$$

Show that componentwise addition and multiplication turn $\left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}}$ into a commutative ring. (4P)

Problem 3: Define two elements $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}}$ as *equivalent*, in signs $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|x_n - y_n| < \varepsilon \quad \text{for all } n \geq N \ .$$

(a) Show that $\sim$ is an equivalence relation. Denote the equivalence class of an element $(x_n)_{n\in\mathbb{N}} \in \left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}}$ by $[(x_n)_{n\in\mathbb{N}}]$. (2P)

(b) Define an equivalence class $[(x_n)_{n\in\mathbb{N}}]$ as *positive*, if there exists a rational $c > 0$ and an $N \in \mathbb{N}$ such that $x_n \geq c$ for all $n \geq N$. Prove that for an equivalence class $[(x_n)_{n\in\mathbb{N}}]$ exactly one of the following holds true:

(i) $[(x_n)_{n\in\mathbb{N}}]$ is positive.

(ii) $[(-x_n)_{n\in\mathbb{N}}]$ is positive.

(iii) $[(x_n)_{n\in\mathbb{N}}] = 0$, where 0 is the zero sequence.

(2P)

(c) Define addition and multiplication on the quotient space $\mathbb{R} := \left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}} / \sim$ by the following:

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \left([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]\right) \mapsto [(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}}],$$
$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \left([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]\right) \mapsto [(x_n)_{n\in\mathbb{N}} \cdot (y_n)_{n\in\mathbb{N}}] \ .$$

Show that these operations are well-defined and turn $\mathbb{R}$ into a field. (5P)

(d) Define an order relation on the quotient space $\mathbb{R} := \left(\mathbb{Q}^{\mathbb{N}}\right)_{\mathrm{C}} / \sim$ by

$$[(x_n)_{n\in\mathbb{N}}] \leq [(y_n)_{n\in\mathbb{N}}] \quad \text{iff } [(y_n)_{n\in\mathbb{N}}] - [(x_n)_{n\in\mathbb{N}}] \text{ is positive or } 0 \ .$$

Show that that is a total order on $\mathbb{R}$ indeed. (2P)

(e) Prove that $\mathbb{R}$ is a Dedekind complete ordered field. It is called the *field of real numbers.* (3P)