Math 2002 Number Systems
Homework Set 5

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For the following problems recall that the set $\mathbb{Z}$ of integers is defined as the quotient set $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$, where $\sim$ is the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined as follows:

$$(n, m) \sim (\tilde{n}, \tilde{m}) \iff n + \tilde{m} = \tilde{n} + m \quad \text{where } n, \tilde{n}, m, \tilde{m} \in \mathbb{N} .$$

Recall further that $[n, m]$ denotes the equivalence class of the pair (n, m). Addition on $\mathbb{Z}$ is then defined by

$$+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \ \big([n, m], [k, l]\big) \mapsto [n + k, m + l] \ ,$$

and multiplication by

$$\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \ \big([n, m], [k, l]\big) \mapsto [n \cdot k + m \cdot l, m \cdot k + n \cdot l] \ .$$

Problem 1: Verify the following properties of addition and multiplication in $\mathbb{Z}$:

(a) associativity of addition, (2P)

(b) commutativity of addition, (1P)

(c) additive neutrality of $0 = [0, 0]$, (1P)

(d) existence of additive inverses, (1P)

(e) associativity of multiplication, (3P)

(f) commutativity of multiplication, (1P)

(g) multiplicative neutrality of $1 = [1, 0]$, (1P)

(h) distributivity of multiplication over addition. (3P)

Problem 2: Define an order relation on $\mathbb{Z}$ as follows:

$$p \leq q \iff \exists n \in \mathbb{N} : \ p + n = q \ .$$

Verify that $\leq$ is an order relation on $\mathbb{Z}$ indeed and that it satisfies the following monotony laws, where p, q are always integers:

Monotony of addition

If $p \leq q$ and $r \in \mathbb{Z}$, then $p + r \leq q + r$.

Monotony of multiplication

If $p \leq q$ and $r \in \mathbb{N}$, then $p \cdot n \leq q \cdot n$.

(6P)

Problem 3:

(a) Which elements in $\mathbb{Z}$ do have a multiplicative inverse?

(b) Verify that 0 annihilates $\mathbb{Z}$ that is that $0 \cdot p = p \cdot 0 = 0$ for all $p \in \mathbb{Z}$.

(c) Show that $(-p) \cdot (-q) = p \cdot q$, where $p, q \in \mathbb{Z}$ and $-p$ and $-q$ denote the additive inverses of p and q, respectively.

(9P)