

Math 2002 Number Systems
Homework Set 5

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For the following problems recall that the set \mathbb{Z} of integers is defined as the quotient set $\mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \sim$, where \sim is the equivalence relation on $\mathbb{N} \times \mathbb{N}$ defined as follows:

$$(n, m) \sim (\tilde{n}, \tilde{m}) \iff n + \tilde{m} = \tilde{n} + m \quad \text{where } n, \tilde{n}, m, \tilde{m} \in \mathbb{N} .$$

Recall further that $[n, m]$ denotes the equivalence class of the pair (n, m) . Addition on \mathbb{Z} is then defined by

$$+ : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}, ([n, m], [k, l]) \mapsto [n + k, m + l] ,$$

and multiplication by

$$\cdot : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}, ([n, m], [k, l]) \mapsto [n \cdot k + m \cdot l, m \cdot k + n \cdot l] .$$

Problem 1: Verify the following properties of addition and multiplication in \mathbb{Z} :

- (a) associativity of addition,
- (b) commutativity of addition,
- (c) additive neutrality of $0 = [0, 0]$,
- (d) existence of additive inverses,
- (e) associativity of multiplication,
- (f) commutativity of multiplication,
- (g) multiplicative neutrality of $1 = [1, 0]$,
- (h) distributivity of multiplication over addition.

(8P)

Problem 2: Define an order relation on \mathbb{Z} as follows:

$$p \leq q \iff \exists n \in \mathbb{N} : p + n = q .$$

Verify that \leq is an order relation on \mathbb{Z} indeed and that it satisfies the following monotony laws, where p, q are always integers:

Monotony of addition

If $p \leq q$ and $r \in \mathbb{Z}$, then $p + r \leq q + r$.

Monotony of multiplication

If $p \leq q$ and $r \in \mathbb{N}$, then $p \cdot r \leq q \cdot r$.

(6P)

Problem 3:

- (a) Which elements in \mathbb{Z} do have a multiplicative inverse?
- (b) Verify that 0 annihilates \mathbb{Z} that is that $0 \cdot p = p \cdot 0 = 0$ for all $p \in \mathbb{Z}$.
- (c) Show that $(-p) \cdot (-q) = p \cdot q$, where $p, q \in \mathbb{Z}$ and $-p$ and $-q$ denote the additive inverses of p and q , respectively.

(6P)