Math 2002 Number Systems
Homework Set 3

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Problem 1: Prove the following statements for all positive natural numbers:

a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$,

b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(6P)

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following claims:

a) If f and g are injective, then $g \circ f$ is injective as well.

b) If f and g are surjective, then $g \circ f$ is surjective, too.

(4P)

Problem 3: Let M be a set and consider its power set $\mathcal{P}M$ with the order relation given by inclusion of sets. Show that $\mathcal{P}M$ has a greatest and a smallest element. Are the greatest and smallest elements uniquely determined? (2P)

Problem 4: Let $p \in \mathbb{N}_{>0}$ denote a positive natural number. Call two integers $m, n \in \mathbb{Z}$ *congruent modulo* p, if p divides $m - n$ that is if there exists $k \in \mathbb{Z}$ such that $m - n = kp$. If m is congruent n modulo p one denotes this by $m \equiv n \mod p$. Show that congruence module p is an equivalence relation on the set of integers $\mathbb{Z}$. Prove also that if

$$m \equiv n \mod p \quad \text{and} \quad m' \equiv n' \mod p \ ,$$

then

$$m + m' \equiv n + n' \mod p \quad \text{and} \quad m \cdot m' \equiv n \cdot n' \mod p \ .$$

(4P)

Problem 5: Let M_1, M_2, N be sets. Show that

(a) $(M_1 \cap M_2) \times N = (M_1 \times N) \cap (M_2 \times N)$ and

(b) $(M_1 \setminus M_2) \times N = (M_1 \times N) \setminus (M_2 \times N)$.

(4P)