MATH 6260 Geometry Quantum Fields
Homework

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Problem 1: Let P be a non-empty set of seminorms on a vector space V. Show that the coarstes translation invariant topology on V such that each $p \in P$ is continuous is a vector space topology on V.

Problem 2: Given a relatively compact open subset $U \subset \mathbb{R}^n$ define the space $\mathcal{C}^N(\overline{U})$ as the space of N-times continuously differentiable functions $f : U \to \mathbb{R}$ such that all partial derivatives $\partial^\alpha f$ up to order $|\alpha| \leq N$ have continuous extensions to $\overline{U}$. For compact $K \subset \overline{U}$ put

$$p_{N,K}(f) = \sum_{\alpha \in \mathbb{N}^n,\, |\alpha| \leq N} \sup_{x \in K} \left| \partial^\alpha f(x) \right| .$$

Show that each of the $p_{N,K}$ is a seminorm and that $\mathcal{C}^N(\overline{U})$ endowed with the norm $p_{N,\overline{U}}$ is a Banach space.

Problem 3: Endow $\mathcal{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ with a family of seminorms such that the resulting locally convex topology has the following property:
A sequence $(f_k)_{k \in \mathbb{N}}$ in $\mathcal{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ converges to some $f \in \mathcal{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ if and only if there exists a compact set $K \subset \mathbb{R}^n$ such that $\mathrm{supp} f_k \subset K$ for all $k \in \mathbb{N}$, $\mathrm{supp} f \subset K$ and such that

$$\lim_{m \to \infty} p_{N,K}(f - f_m) = 0 \quad \text{for all } N \in \mathbb{N} .$$

Problem 4:
Consider the (reduced) Kepler problem. Recall that it has state space $\mathbb{R}^6$ with canonical coordinates $(q^1, q^2, q^3, p^1, p^2, p^3)$ and Hamiltonian $h = \frac{1}{2m}\|p\|^2 - \frac{k}{\|q\|}$, where m is the (reduced) mass and k is a constant. Consider the angular momentum

$$L = q \times p$$

and the Lenz–Runge vector

$$A = p \times L - mk \frac{q}{\|q\|} .$$

(a) Show that the Poisson bracket of the Hamiltonian with each component of the angular momentum and with $\|L\|^2$ vanishes.

(b) Prove that the Lenz-Runge vector is a preserved quantity of the Kepler problem that means that for every solution $\gamma : I \to \mathbb{R}^6$ of Hamilton's equations of the Kepler problem $A \circ \gamma$ is constant.

Problem 5: Let H be a complex Hilbert space. Define for every operator $A \in \mathcal{B}(\mathrm{H})$ the *resolvent set* $\varrho(A)$ by

$$\varrho(A) := \{\lambda \in \mathbb{C} \mid (A - \lambda) \in \mathsf{GL}(\mathrm{H})\}$$

and the *resolvent* $R_\bullet(A) : \varrho(A) \to \mathcal{B}(\mathrm{H})$ by

$$R_\lambda(A) := (A - \lambda)^{-1} \quad \text{for } \lambda \in \varrho(A) \ .$$

The complement $\sigma(A) := \mathbb{C} \setminus \varrho(A)$ then is called the *spectrum* of A. Show the following for the resolvent and the spectrum of A:

(i) $\varrho(A)$ is open in $\mathbb{C}$. More precisely, if $\lambda_0 \in \varrho(A)$, then the open disc $B_{\|R_{\lambda_0}(A)\|^{-1}}(\lambda_0)$ is contained in the resolvent set and

$$R_\lambda(A) = R_{\lambda_0}(A) + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}^{n+1}(A) \quad \text{for } |\lambda - \lambda_0| < \|R_{\lambda_0}(A)\|^{-1} \ ,$$

where convergence is with respect to the operator norm.

(ii) $\sigma(A)$ is compact, more precisely $\sigma(A) \subset \overline{B}_{\|A\|}(0)$, and

$$R_\lambda(A) = -\frac{1}{\lambda} - \sum_{n=1}^{\infty} \lambda^{-n-1} A^n \quad \text{for } |\lambda| > \|A\| \ .$$

(iii) The operators A and $R_\lambda(A)$ commute for all $\lambda \in \varrho(A)$.

(iv) The following resolvent formula holds for all $\lambda, \mu \in \varrho(A)$:

$$R_\mu(A) - R_\lambda(A) = (\mu - \lambda) R_\mu(A) \cdot R_\lambda(A) \ .$$

In particular this means that $R_\mu(A)$ and $R_\lambda(A)$ commute as well.

(v) The resolvent $R_\bullet(A) : \varrho(A) \to \mathcal{B}(\mathrm{H})$ is continuous.

(vi) The resolvent $R_\bullet(A) : \varrho(A) \to \mathcal{B}(\mathrm{H})$ is complex differentiable. More precisely

$$R'_\lambda(A) = \lim_{\mu \to \lambda} \frac{R_\mu - R_\lambda(A)}{\lambda - \mu} = R_\lambda(A)^2 \ .$$

(vii) One has

$$\lim_{|\lambda| \to \infty} \lambda R_\lambda(A) = -\mathrm{id}_{\mathrm{H}} \ .$$

In particular this means $\lim_{|\lambda| \to \infty} R_\lambda(A) = 0$.

(viii) The map $\langle R_\bullet(A) v, w \rangle : \varrho(A) \to \mathbb{C}$ is holomorphic for all $v, w \in \mathrm{H}$.

(ix) $\sigma(A) \neq \emptyset$.

Hint: For the last claim use Liouville's Theorem.